Principles of Time and Space

New Octonions and Relativity

Third Edition

Hiroshige Goto

$$h^2 = i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k, \quad jk = -kj = i$$

$$ki = -ik = j$$

$$hi = ih, \quad hj = jh, \quad hk = kh$$

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This book is dedicated to my beloved wife Rieko

Preface to the Revised Edition

The quaternion is a complex number that extends into four dimensions. In this book, the quaternion is transformed into a number in the curved four-dimensional space–time and the transformed quaternion is called the new octonion. Because the new octonion consists of four real numbers and four imaginary numbers, four kinds are thought to exist (similar to the real and imaginary numbers), each in our space–time. In addition, if the structure of the four-dimensional space–time is examined using the new octonion, space is considered to have a double structure. In other words, two four-dimensional space–time structures overlap. Furthermore, when using the new octonion, the conclusions of special relativity can be explained without contradiction if the axiom that mass is the time component of the unit world line is imposed. This leads to the conclusion that in our world, mass and energy are expressed as imaginary numbers.

An algebraic theorem states that a numerical system with the algebraic operations of addition, subtraction, multiplication, and division is composed only of real numbers, complex numbers, quaternions, and octonions. Therefore, it is natural to assume that the physical laws of four-dimensional space–time must be described with quaternions. William Rowan Hamilton, who discovered the quaternion in 1843, dedicated the latter half of his life to applying quaternions to physics but without success. However, the concept of vectors came to light through his efforts; the concept of vector led to the development of the concept of tensors. Tensors are indispensable to mathematics in modern physics and engineering; in general relativity, black holes were predicted and described using tensors.

Because the tensor is a mathematical tool by which curved four-dimensional space–time is described, it is widely thought that using the quaternion to study space–time would represent a backward motion—an unnecessary and fruitless endeavor. However, tensor has a complex matrix form. The purpose of natural science is to obtain simple laws from complex phenomena. I feel that the usage of a tensor is significantly complex for describing physical laws. This study starts by proposing that a simple algebra such as the quaternion may indeed be suitable to describe the laws of space–time. Therefore, I have tried to create an algebra using only addition, subtraction, multiplication, and division to study curved four-dimensional space–time.

Hamilton did not succeed in applying quaternions to physics because the theory of relativity did not exist at that time. Hamilton's quaternion is the mathematics of flat space–time; however, space–time curves in the theory of relativity. Physical laws must be described with mathematics that matches the curved space–time. In other words, if the quaternion is not altered to suit the demands of the curved space–time, physical laws will not be described by quaternions.

First, the complex number is transformed to fit the mathematics of Lorentz transformations and is called the new complex number. Similar to the complex number, the quaternion is transformed into the curved space–time quaternion. This is dubbed an octonion because it includes eight numbers. The Graves' octonion consists of one real number and seven imaginary numbers, but the octonion of the curved space–time consists of four real numbers and four imaginary numbers. Therefore, it is termed a new octonion. If the space–time structure is examined with the new octonion and the mechanics of special relativity is recalculated, new interpretations of the physical principles and double structure of space–time are suggested.

However, the space–time diagram suggested by the new octonion may disagree with actual space–time even if the new octonion is mathematically consistent. It is well known that we cannot use Euclidean geometry, which is consistent in flat space, to study curved space–time. The actual space–time may not have a double structure even if it is suggested by the new octonion. Future problems include whether the new octonion is mathematics without contradiction and whether the actual space-time is identical to that suggested by the new octonion.

The desire to learn the truth is a motivational force of science. Steven Paul Jobs, one of the founders of Apple Computer, Inc., was quoted as saying "Stay hungry, stay foolish," which means that we should always seek better things and not be tethered to preconceived notions. The physics described by tensors is accomplished. However, it is a scientific action to seek a different mathematics to describe curved space–time.

Because the new octonion is simple mathematics, it can be understood even by a high school student. It is my wish that people who are interested in physics and mathematics and who want to learn the truth must acknowledge this book and use the new octonion in their studies.

Hiroshige Goto

August 2012

Abstract

The contents of this book are summarized here. Because the technical terms are explained in the text, they are used here without explanation.

Complex numbers have been expanded to Hamilton's quaternion in four dimensions. This study aims to transform the quaternion into mathematics suitable for describing curved four-dimensional space–time. An algebraic theorem states that a numerical system with the algebraic operations of addition, subtraction, multiplication, and division has only real numbers, complex numbers, quaternions, and octonions. Therefore, the physical principles of four-dimensional space–time should be describable using quaternions.

First, it is found that the division

$$\overline{A}/\left|A\right|$$

of a complex number is the coordinate transformation if A is a complex number and $\overline{A}$ is its complex conjugate. (**Chap. 2**)

Therefore, it is considered that the Lorentz transformations for special relativity

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}},$$
$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

can be obtained if $\overline{A}/\left|A\right|$ is applied in two-dimensional space–time. The result is that the sign of the velocity of light squared of the Lorentz transformations changes from minus to plus, presumably because complex numbers are the mathematics of flat space–time. However, the space–time is curved. Therefore, new complex numbers must be created to describe the curved space–time. It is assumed that a fourth imaginary number h exists in addition to the three imaginary numbers $i,\ j,\ k$ of Hamilton's quaternions. The algorithms are as follows:

$$i^2 = j^2 = k^2 = h^2 = -1,$$

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,$$

$$hi = ih,\ hj = jh,\ hk = kh.$$

A new complex number $ah + bi$ is created from the complex number $a + bi$ and equations identical to the Lorentz transformations are obtained when the coordinate transformation $\overline{A}/|A|$ is done with this new complex number. This shows that the new complex number gives mathematics of the curved space–time. (**Chap. 3**)

In addition, it is considered that time advances only in the positive direction because a contradiction occurs if time t is assumed as negative when we use the coordinate transformation $\overline{A}/|A|$ to obtain the Lorentz transformations. (**Chap. 3**)

Herein, we examine the new complex number to determine whether it leads to accurate mathematics. Time dilation, proper time, length contraction, world distance, and the twin paradox of special relativity in two-dimensional space–time are proven with the new complex numbers and new complex planes. Without light emission and examples of a clock and ruler, results identical to those of special relativity are obtained using only the new complex number and the equation of the world line. (**Chap. 6, Chap. 7, Chap. 8, Chap. 9**)

Next, by using the fourth imaginary number h, a new quaternion $ah+bi+cj+dk$ in the curved space–time is created from Hamilton's quaternion $a + bi + cj + dk$. Furthermore, the new quaternion is used in the coordinate transformation $\overline{A}/|A|$ and the Lorentz transformations in the curved four-dimensional space–time are obtained. These equations are called the new Lorentz transformations. If observer B moves with constant velocity v in the x-direction with respect to observer A at rest,

$$y' = y,$$
$$z' = z$$

in special relativity. However, y' and z' are functions of the three variables y, z, and v, and the x motion changes the y and z distances if these are calculated using the new quaternion mathematics without assuming the isotropy of space–time. These equations are as follows:

$$y' = \frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}},$$
$$z' = \frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}}.$$

Because of the homogeneity of space and the fact that $y' = f(v)y$ and $z' = g(v)z$ were assumed in special relativity, these results were not obtained. Because a constant velocity of light is obtained by the velocity-transformation equations and because the world distance calculated by these transformations is invariant, the accuracy of the new Lorentz transformations is proven. (**Chap. 10**)

In addition, unlike the definition of world distance of special relativity, i.e.,

$$s^2 = (ct)^2 - x^2 - y^2 - z^2,$$

the definition of world distance calculated by the new quaternion is

$$s^2 = -(ct)^2 + x^2 + y^2 + z^2. \qquad (\textbf{Chap.10})$$

Here, we prove that the new quaternion has eight numbers that consist of a real number a, imaginary numbers h, i, j, k, and products hi, hj, hk of two imaginary numbers. We also show that hi, hj, and hk are new real numbers that are independent of other numbers. In other words, the new quaternion is the octonion that consists of four real numbers and four imaginary numbers. This is called the new octonion because it is the mathematics of curved space–time unlike Graves' octonion (Cayley number) in flat space, which is already known. Cayley numbers consist of one real number and seven imaginary numbers. (**Chap. 11**)

The space–time structure is examined using the new octonion. This examination suggests that four-dimensional space–time has a double structure. We live in a world where the world point is expressed as $cth + xi + yj + zk$ and the number of the world point in an alternative world is $ct + xhi + yhj + zhk$. Strictly speaking, our world is a world of imaginary numbers, while the other world is that of real numbers. However, our world is called the positive world and the other world is called the negative world until a time when this space–time theory becomes more prevalent. The positive and negative worlds do not exist in parallel but rather overlap. In other words, each coordinate axis of the four-dimensional space–time has a coordinate that is part of the positive world as well as one that is part of the negative world; the temporal-axis parts are ct_0h and ct_1, the x-axis parts are x_0i and x_1hi, the y-axis parts are y_0j and y_1hj, and the z-axis parts are z_0k and z_1hk. If we assume only one four-dimensional space-time, each coordinate part of the four-dimensional space–time is a complex number. The temporal axis part is $ct_0h + ct_1$, the x-axis part is $(x_0 + x_1h)i$, the y-axis part is $(y_0 + y_1h)j$, and the z-axis part is $(z_0 + z_1h)k$. (**Chap. 11**)

In addition, by using the number $ct+xhi+yhj+zhk$ to express the world point in the negative world, the Lorentz transformations in the negative world are obtained. The results indicate that the physical laws in the negative world are the same as those in the positive world. (**Chap. 11**)

Furthermore, it is proven that the coordinate transformation $\overline{A}/|A|$ creates oblique coordinate axes. (**Chap. 12**)

To examine whether the new octonion is correct, its axioms and theorems are chosen. It is sufficient to examine the erratum of axioms and theorems to see whether the new octonion causes a contradiction. In addition, it is known that the associative law

$$(AB)C = A(BC)$$

is not possible with Graves' octonion, but it is proven to be possible with the new octonion. Based on this, the new octonion is thought to be a more complete number than Graves' octonion, which has been used in a recent study of fundamental particles. However, researchers may not obtain precise conclusions without the new octonion. (**Chap. 13**)

In addition, because the new octonion space–time is bent but parallel lines do not cross there, the new octonion geometry is a non-Euclidean geometry and non-Riemannian geometry, respectively. (**Chap. 13**)

We can rewrite a vector with the new octonion if we accept the double structure of four-dimensional space–time. If we assume that there are two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ and new octonions A and B express each vector, the relationships between the new quaternion, scalar product, and vector product are as follows:

$$B\overline{A} = \boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A}\times\boldsymbol{B},$$
$$\boldsymbol{A}\cdot\boldsymbol{B} = (B\overline{A} + A\overline{B})/2,$$
$$\boldsymbol{A}\times\boldsymbol{B} = (B\overline{A} - A\overline{B})/2.$$

Furthermore, triple and quadruple vector products can be rewritten using the new octonion. (**Chap. 14**)

In addition, the metric tensor and Kronecker δ are not required because these tensors can be calculated with the new octonion in straight coordinate systems. The differences between a tensor and the new octonion are examined. Tensor mathematics calculates physical quantities in four-dimensional space–time by confining them to a cross section of four-dimensional space–time. In contrast, the new octonion mathematics calculates physical quantities throughout the entire four-dimensional space–time. (**Chap. 15**)

Synchrotron radiation is also explained with the new octonion. According to calculations based on Lorentz transformations, the light laterally released by an electron moving with the velocity of light angles forward, but using the new Lorentz transformations, it becomes bidirectional light, i.e., a combination of lights in the front and lateral directions. At present, the interpretation of this result is unknown. (**Chap. 16**)

In addition, momentum conservation and the change of mass due to velocity effects under the new Lorentz transformations are proven. That these results are identical to the conclusions of special relativity shows that the new Lorentz transformations found by the new octonion are correct. (**Chap. 17**)

Furthermore, it is shown that the results of special relativity can be explained without contradiction if mass and energy are viewed as the time part of the unit world line and momentum as the space parts of the unit world line. Unlike the result of general relativity, mass does not bend space–time and the time part of the unit world line of a particle itself is mass. In addition, mass and energy in the positive world are described with imaginary numbers if we assume that mass is the time part of the unit world line. At that time, mass and energy in the negative world are described with real numbers and the sum of the squares of mass or energy in the positive and negative worlds becomes zero. In general, it is thought that the addition of mass or energy of matters and antimatters yields zero. However, this is not correct. (**Chap. 18**)

Because the world distance of a point on $x = ct$ is zero, light possesses no rest mass. The world distance of the point on $x = vt$, on the other hand, is non-zero; hence, substance has mass. That is, mass is not given; however, it is finite or zero depending on the locus in four-dimensional space–time. In this picture, no Higgs boson is required. (**Chap. 18**)

It is proven that the energy conservation law and the momentum conservation law are laws of reflection of world lines. This is proven under the assumption that force does not exist and the curve of the world line by collision and reflection of the world lines is force. (**Chap. 18**)

In addition, Hiroyuki Kamada demonstrated that Dirac's γ matrix and the new octonion are mathematically equivalent (personal communication). (**Chap. 18**)

Although it is not the original purpose of this book, the new Lorentz transformations with superluminal velocity is found by coordinate transformations using the new octonion. The result shows that a particle with superluminal velocity is in the negative world and we cannot observe it from the positive world. In addition, time cannot be reversed from the equation of proper time. In other words, we cannot return to the past even by moving at superluminal velocity. It is also shown that space–time is discontinuous. (**Chap. 19**)

Other problems related to space–time theories that can be solved by the new octonion are also shown. Maxwell's equations of electromagnetic waves satisfy the Lorentz transformations. However, how do they change if the new Lorentz transformations are applied? Can we find negative-world parts of the electromagnetic wave like the y'- and z'-axis parts of the new Lorentz transformations? In addition, if it is assumed that differences between the world lines of light and matter are only differences between the places of passage in four-dimensional space–time, the world line of a point mass has the property of a wave. In that case, what will the Maxwell's equations of a matter wave be? Is there a possibility that the equations lead to the unified field theory? (**Chap. 20**)

It is revealed that biquaternion, which was discovered by Hamilton in 1844, and the new octonion are equivalent. However, their interpretation and application methods markedly differ. In the biquaternion, h is treated as an attached imaginary number rather than a fourth imaginary number. On the other hand, in the new octonion, h is the fourth imaginary number and is more important than i, j, and k. The important physical quantity of special relativity (proper time) is denoted by the imaginary number h. Mass and energy are also related to h. (**Chap. 20**)

Although the velocity of light is not constant in an accelerated system in general relativity, can we obtain an identical conclusion using the new octonion? The Lorentz transformations in an accelerated system do not exist, but equations equivalent to the new Lorentz transformations can be obtained by coordinate transformations using the new octonion. The results of such calculations are shown. (**Chap. 20**)

Finally, it is shown that the new octonion and the current string theory may have identical contents. According to the space–time theory using the new octonion, all fundamental particles are constrained by five variables: (ct, x, y, z) express the space–time position in four dimensions and the variable Ψ (psi) expresses the amplitude of the world line. In addition, the size of Ψ is smaller than that of a fundamental particle because Ψ is the thickness of the world line. In this reckoning, there is something akin to the five dimensions of the space–time proposed by Kaluza–Klein theory, which precedes string theory. In that theory, the fifth dimension has the shape of a ring that is too miniscule to be observed. This fifth dimension may be identical to the amplitude Ψ of the world line. Because all particles have five variables (ct, x, y, z, Ψ), five dimensions of space–time follow easily. However, only four-dimensional space–time exists because Ψ is an amplitude. In addition, because the positive and negative worlds have an overlapping double structure, there are ten variables between the two worlds—(ct, x, y, z, Ψ) for the positive world and (ct', x', y', z', Ψ') for the negative world. Because the string theory developed from Kaluza–Klein theory exists only in ten-dimensional space–time, proponents of the theory may think that there also must be ten dimensions in the world. However, only four-dimensional space–time with a double structure exists. Although all particles are expressed with oscillating strings by string theory, the world line of a particle vibrates in a way similar to light according to space–time theory using the new octonion. The smallest unit of the world line is smaller than a fundamental particle from a space–time discontinuity. Because space–time theory using the new octonion and string theory may have the same contents, the problem is whether we can rewrite string theory with the new octonion. (**Chap. 20**)

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Contents

1

Basic Knowledge of Complex Numbers

1.1 Definition and formulae of complex numbers

In this book, the complex number and the quaternion are changed into mathematical entities suitable for curved four-dimensional space–time: new complex numbers, new quaternions, and new octonions. There may be readers who think they cannot understand the contents of this book if they have not studied the quaternion and the octonion, or even the complex number. However, only the basic knowledge of the complex number is enough to understand the contents of this book, and necessary components are explained in this chapter.

A complex number is the combination of a real number and an imaginary number. We define the imaginary number i as a number that, when squared, becomes -1, or $i^2 = -1$. This is a departure from our common experience because when a number that is used every day is squared, it does not become negative, even if it itself is a negative number. This type of number is called a real number. Putting them together yields a complex number. For example, if a and b are real numbers, a complex number A can be represented as $A = a + bi$. We call a the real part and b the imaginary part.

Typically, the imaginary unit is written at the beginning of a term as ib. However, in this book, the order is reversed so that a number is followed by the imaginary unit, i.e., bi. This practice serves the customary formalism enhances comprehension.

Given two complex numbers $A = a + bi$ and $B = c + di$, the formulae are

$$\begin{aligned} A + B &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i, \\ A \times B &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \end{aligned}$$

$$= (ac - bd) + (ad + bc)i.$$

$A \div B$ is explained later in this section. In addition, if $A = B$, we can write $a = c,\ b = d$, which we call as coefficient comparison.

Concerning complex numbers $A,\ B$, and C, we have natural results:

$$A + B = B + A,$$

$$AB = BA,$$

$$(A + B) + C = A + (B + C),$$

$$(AB)C = A(BC),$$

$$A(B + C) = AB + AC.$$

As a quick look ahead, note that in Hamilton's quaternion, which will be explained later, the commutative property of multiplication (second equation from above) does not hold. That is

$$AB \neq BA.$$

In other words, multiplication order matters in Hamilton's quaternion.

If complex number is $A = a + bi$, then its complex conjugate is defined as $\overline{A} = a - bi$. To make a complex conjugate out of a complex number, simply switch the sign on the imaginary part. We call the size of A as the magnitude of A and write it as $|A|$. Furthermore, we find

$$\begin{aligned} |A|^2 &= A\overline{A} \\ &= (a + bi)(a - bi) \\ &= a^2 - abi + bai - b^2 i^2 \\ &= a^2 + b^2. \end{aligned} \tag{1.1}$$

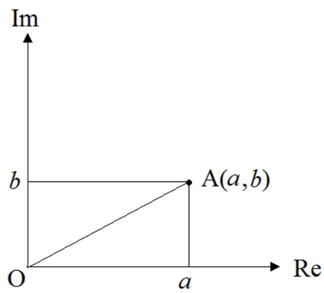


Figure 1.1

To represent complex numbers on a two-dimensional coordinate grid, the horizontal axis is taken to be the real axis and the vertical axis is the imaginary axis, as in Figure 1.1. Then, the real axial part of a complex number $A = a + bi$ is a and the imaginary axial part is b. A is expressed as the point $(a,\ b)$. In addition, $|A|$ expresses the length from origin O to the point A. A formula

$$|A|^2 = a^2 + b^2$$

is obvious by the Pythagorean theorem and we can clearly see that the definition $|A|^2 = A\overline{A}$ in (1.1) is correct.

Because division $1/A$ is calculated by multiplying the numerator and denominator by $\overline{A}$, we have

$$\begin{aligned} \frac{1}{A} &= \frac{\overline{A}}{A\overline{A}} \\ &= \frac{\overline{A}}{|A|^2}. \end{aligned} \tag{1.2}$$

Thus, if $B = c + di$ and $B \neq 0$, the equations are

$$\begin{aligned} A \div B &= \frac{A}{B} \\ &= \frac{A\overline{B}}{B\overline{B}} \\ &= \frac{A\overline{B}}{|B|^2} \\ &= \frac{(a+bi)(c-di)}{c^2+d^2} \\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}. \end{aligned}$$

In addition, we can also write

$$\begin{aligned} \overline{A+B} &= \overline{A} + \overline{B}, \\ \overline{AB} &= \overline{A}\ \overline{B}. \end{aligned}$$

However, because the placement of $\overline{A}$ and $\overline{B}$ is reversed in the quaternion and the octonion, the equation becomes

$$\overline{AB} = \overline{B}\ \overline{A}.$$

With complex numbers,

$$\overline{A}\ \overline{B} = \overline{B}\ \overline{A}.$$

Thus, if we write

$$\overline{AB} = \overline{B}\ \overline{A},$$

even in the complex number as in the quaternion, the new quaternion, and the octonion, there may be no confusion. We will explain this in more detail in Theorem 3 of Section 13.3.

1.2 Complex numbers and rotation

In this section, we explain the relation between the complex number and rotation. This is rarely taught in high schools, but we can explore it here using simple examples.

First, we consider multiplication. Figure 1.2 is drawn assuming that $A = \sqrt{3} + i,\ B = 2 + 2\sqrt{3}i$.

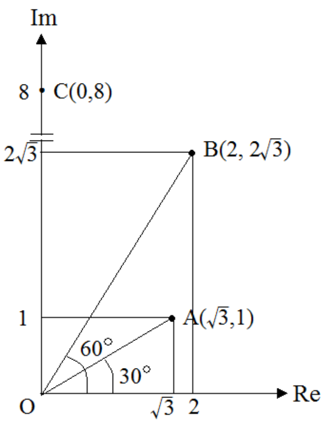


Figure 1.2

From (1.1), the magnitudes squared of A and B are

$$\begin{aligned}
|A|^2 &= (\sqrt{3} + i)(\sqrt{3} - i) \\
&= (\sqrt{3})^2 - i^2 \\
&= 3 + 1 \\
&= 4, \\
|B|^2 &= (2 + 2\sqrt{3}i)(2 - 2\sqrt{3}i) \\
&= 4 - (2\sqrt{3}i)^2 \\
&= 4 + 12 \\
&= 16.
\end{aligned}$$

Thus, the magnitudes of A and B are

$$|A| = 2,\ \ |B| = 4.$$

The angle that the real axis makes with the line segment OA or OB is called the argument. The arguments of A and B are 30° and 60°, respectively, with each value arrived at using the ratios of triangular sides. If $A \times B$ is calculated, multiplying gives

$$\begin{aligned} A \times B &= (\sqrt{3} + i)(2 + 2\sqrt{3}i) \\ &= 2\sqrt{3} + 2(\sqrt{3})^2 i + 2i + 2\sqrt{3}i^2 \\ &= 2\sqrt{3} + 6i + 2i - 2\sqrt{3} \\ &= 8i. \end{aligned}$$

Because $A \times B$ has only an imaginary part, it solely lies on the imaginary axis and its argument is 90°. This argument of 90° means the addition of the 30° argument of A and the 60° argument of B. In addition, magnitude 8 of $A \times B$ is the multiplication of magnitude 2 of A and magnitude 4 of B.

As we infer from this example, we have in general

$$\begin{aligned} arg(A \times B) = argA + argB, \\ |A \times B| = |A| \times |B|. \end{aligned}$$

In other words, we have $A \times B$ if we rotate A counterclockwise on origin O by the argument of B and multiply the magnitude of A by the magnitude of B.

Next, we consider division. Given $A = \sqrt{3} + i$ and $C = 8i$, from (1.2), $C \div A$ is

$$\begin{aligned} C \div A &= \frac{C}{A} \\ &= \frac{C\overline{A}}{A\overline{A}} \\ &= \frac{8i(\sqrt{3} - i)}{(\sqrt{3} + i)(\sqrt{3} - i)} \\ &= \frac{8\sqrt{3}i - 8i^2}{3 - i^2} \\ &= \frac{8 + 8\sqrt{3}i}{4} \\ &= 2 + 2\sqrt{3}i. \end{aligned}$$

Thus, $C \div A$ is identical to B. Because the magnitude squared of $C \div A$ is

$$|C \div A|^2 = (2 + 2\sqrt{3}i)(2 - 2\sqrt{3}i)$$

$$= 4 - 4(\sqrt{3})^2 i^2$$
$$= 4 + 12$$
$$= 16,$$

we have

$$|C \div A| = 4.$$

We can draw Figure 1.2.

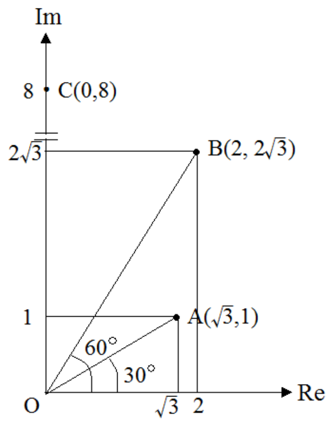


Figure 1.2

By the ratio of the sides, the argument of $C \div A$ is 60°. It is the number that we have if an argument of 30° of A is subtractd from an argument of 90° of C. Furthermore, the magnitude of $C \div A$ is the number that we have if magnitude 8 of C is divided by magnitude 2 of A.

As we infer from this example, we have in general

$$arg(C \div A) = argC - argA,$$
$$|C \div A| = |C| \div |A| \, .$$

In other words, we have $C \div A$ if we rotate C clockwise on origin O by the argument of A and divide the magnitude of C by the magnitude of A.

We now see that multiplication and division of complex numbers are rotations on origin O. We add the arguments if we multiply and subtract them if we divide.

We can generally discuss the rotations using the trigonometric function $\sin\theta$ (theta) and $\cos\theta$. However, it is unknown as to whether angle θ changes by rotation in curved space–time. Therefore, $\sin\theta$ and $\cos\theta$ are not used in this book. This is explained in Theorem 10 of Section 13.3.

2

Coordinate Transformation by Complex Numbers

2.1 About coordinate transformation

It is necessary to make a general formula of coordinate transformation by a complex number to find the Lorentz transformations of special relativity using complex numbers. First, we explain coordinate transformation. We consider a case in which two observers P and Q observe a certain point R. We assume that P is at rest and Q moves. The time t and position x of R observed by P will be different from those observed by Q. Because coordinate means how to be seen, coordinates $(t,\ x)$ of R observed by P and coordinates $(t',\ x')$ of R observed by Q must be different.

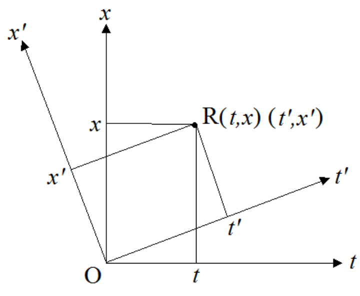


Figure 2.1

The horizontal axis is defined as the temporal axis and the vertical axis is defined as the length axis in a two-dimensional coordinate plane, as shown in Figure 2.1. If the position of R is fixed, each coordinate part is the coordinate of the point where the perpendicular lines drawn to each coordinate axis from R cross with that coordinate axis. Therefore, the t'-axis and the x'-axis of Q turn around origin O for the t-axis and the x-axis of P.

Because the term is coordinate transformation, the coordinates $(t,\ x)$ of R observed by P are transformed into the coordinates $(t',\ x')$ of R observed by Q. Mathematically, equations to express relations between $(t',\ x')$ and $(t,\ x)$ are the equations of coordinate transformation. Lorentz transformations of special relativity are the equations of coordinate transformation. Though Lorentz transformations were obtained using the constancy of the velocity of light, we will obtain them by coordinate transformations using complex numbers in this book.

A Minkowski coordinate diagram, or Minkowski space–time diagram, uses these principles to describe the coordinate plane of special relativity, in which length x lies on the horizontal axis and time t on the vertical axis. However, in this book, we determine that the horizontal axis will be the temporal axis and the vertical axis will be the length axis. There are advantages of calculations and understandings in this coordinate diagram because the slope of a straight line expresses velocity, unlike that in the case of Minkowski space–time diagrams.

2.2 General formulae of coordinate transformation by complex numbers

We consider a two-dimensional complex plane, such as the one in Figure 2.2.

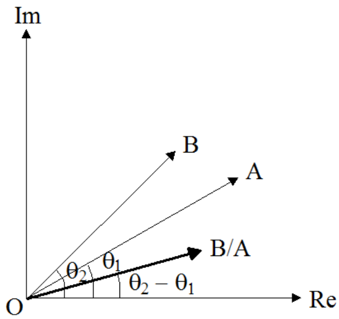


Figure 2.2

We assume that the magnitudes of A and B are $|A|$ and $|B|$, and the angles that complex numbers A and B make with the real axis are $argA = \theta_1$ and $argB = \theta_2$. As for complex number B/A, which is the same as $B \div A$, the magnitude and argument are as follows:

$$\left|\frac{B}{A}\right| = \frac{|B|}{|A|}, \tag{2.1}$$

$$arg(B/A) = argB - argA = \theta_2 - \theta_1, \tag{2.2}$$

from (1.3) and (1.4), as explained in Section 1.2. Because $\angle AOB = \theta_2 - \theta_1$ in Figure 2.2, B/A is piled on B, as shown in Figure 2.3, if the real axis is turned counterclockwise around origin O and is put on A.

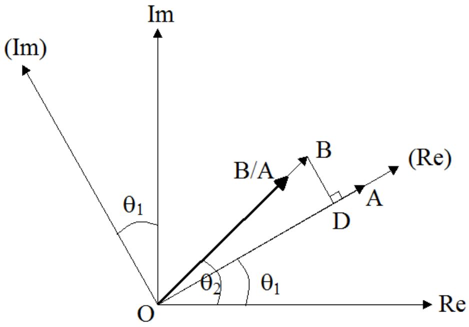


Figure 2.3

The magnitude of $(B/A)\,|A|$, which we find by multiplying the complex number B/A by $|A|$, is

$$\left|\frac{B}{A}\right| |A| = \frac{|B|}{|A|}\,|A|$$
$$= |B|$$

from (2.1). Therefore, $(B/A)\,|A|$ expresses the real number part and the imaginary number part of B if we assume that A is the new real axis. In other words, if we assume that D is the intersection of the perpendicular line drawn to A from B, we can write

$$\frac{B}{A}\,|A| = |OD| + |BD|\,i, \tag{2.3}$$

where i is an imaginary number. $|OD|$ and $|BD|$ express lengths between two points and do not express the magnitudes of complex numbers. From (2.3), we can understand that $(B/A)|A|$ expresses how B is seen by A. Therefore, it is the coordinate transformation of B by A.

There may be readers who think that in Figure 2.3, the i of the imaginary axis before the rotation and the i of the imaginary axis after the rotation are not identical because they are in a different position. However, each i is identical because strictly speaking, the real axis does not rotate; rather, A and B rotate to the real axis. We only rotate the coordinate axis so that it is easy to be understood. Figures are described using this form from here on. In Theorem 9 of Section 13.3, we examine the cases, in which we can perform coefficient comparison and where we cannot.

If we multiply the numerator and the denominator of a fraction by complex conjugate $\overline{A}$ to change an equation of division (2.3) into an equation of the multiplication, we have

$$\begin{aligned}\frac{B}{A}\,|A| &= \frac{B\,|A|\,\overline{A}}{A\overline{A}} \\ &= \frac{B\,|A|\,\overline{A}}{|A|^2} \\ &= \frac{B\overline{A}}{|A|}.\end{aligned}$$

In other words,

$$\frac{B\overline{A}}{|A|} \tag{2.4}$$

is an equation to express the coordinate transformation of B by A.

If observers P, Q, and observed point R in the last section are applied, complex number B expresses the coordinates $(x,\ t)$ of R observed by P at rest. Complex number A expresses the coordinates of moving observer Q and (2.4) expresses the coordinates $(x',\ t')$ of R observed by Q. Relations of $(x,\ t)$ and $(x',\ t')$ are calculated using (2.4) in Section 3.3.

From the above results, if we assume that A is a complex number expressing the coordinates of Q, the equation of the coordinate transformation by moving observer Q is

$$\frac{\overline{A}}{|A|}. \tag{2.5}$$

2.3 Constancy of the physical quantity through coordinate transformation

We explain an important property of (2.5) here. A quantity that does not change in magnitude under Lorentz transformation is called an invariant quantity under coordinate transformation in special relativity. Researchers have defined world distance, four-velocity, and four-momentum as the physical quantities that do not change under coordinate transformation and consider the motion of point mass in four-dimensional space–time. In other words, coordinate transformation that does not change magnitudes is important in physics. From the calculations mentioned above and in Figure 2.3, the coordinate transformation (2.4) by a complex number only changes the coordinate parts of B, i.e., how B is seen, and it does not change the magnitude of B. We will prove this in the following text.

If F is the complex number result of B following coordinate transformation by A, we can write

$$\frac{B\overline{A}}{|A|} = F.$$

Thus, $|F|^2$ is given as follows:

$$\begin{aligned} |F|^2 &= F\overline{F} \\ &= \frac{B\overline{A}}{|A|}\frac{A\overline{B}}{|A|} \\ &= \frac{|A|^2\,|B|^2}{|A|^2} \\ &= |B|^2 . \end{aligned}$$

In other words, because the magnitude of $B\overline{A}/|A|$ is identical to that of B, coordinate transformation $\overline{A}/|A|$ does not change the magnitude of B in the complex plane, which is the length of B from origin O.

2.4 Inverse transformation of coordinate transformation

We find the inverse transformation of $\overline{A}/\,|A|$, which will enable us to change $B\overline{A}/\,|A|$ back into B. If we multiply both sides of $B\overline{A}/\,|A| = F$ by $A/\,|A|$, the equations become

$$\begin{aligned} \frac{B\overline{A}}{|A|}\frac{A}{|A|} &= F\frac{A}{|A|}, \\ \frac{B\,|A|^2}{|A|^2} &= F\frac{A}{|A|}, \\ B &= F\frac{A}{|A|}. \end{aligned}$$

Thus, the inverse transformation of $\overline{A}/\,|A|$ is

$$\frac{A}{|A|}.$$

We can also obtain this by

$$\begin{aligned} \frac{\overline{A}}{|A|}\frac{A}{|A|} &= \frac{|A|^2}{|A|^2} \\ &= 1. \end{aligned}$$

3

Complex Numbers and Lorentz Transformations

3.1 Standard derivation methods of Lorentz transformations

We explain the method to find Lorentz transformations in special relativity before obtaining Lorentz transformations using the coordinate transformation $\overline{A}/|A|$ by complex numbers. This is a standard method that can be found in any book about relativity theory. In the following explanation, the symbol A represents observer A, not the complex number A of $\overline{A}/|A|$ from coordinate transformations.

We assume that the coordinates in four-dimensional space–time of the tip of light released at time $t = 0$ from origin O are $(ct,\ x,\ y,\ z)$ as seen from observer A and $(ct',\ x',\ y',\ z')$ as seen from observer B. The variable c represents the velocity of light, t expresses time, and $x,\ y$, and z express lengths. Because the tip of the light makes a spherical surface of radius ct in the three-dimensional space at time t, the equation is given as follows:

$$x^2 + y^2 + z^2 = (ct)^2. \tag{3.1}$$

A coincides with B at origin O at time $t = 0$. The constancy of the velocity of light means that the velocity of light is c regardless of whether the light is observed by an inertial observer or by an observer at rest. The equation of the spherical surface seen by B is as follows:

$$x'^2 + y'^2 + z'^2 = (ct')^2, \tag{3.2}$$

wherever B is. We have $y = y' = 0$ and $z = z' = 0$ if B moves in the x-direction with respect to observer A at rest with a constant velocity v. Under conditions $x \geqq 0,\ t \geqq 0$, (3.1), and (3.2), we have

$$x = ct,$$

$$x' = ct'.$$

We assume that functions satisfying these two equations using fixed number a are

$$x' = a(x - vt),$$
$$x = a(x' + vt').$$

From the four equations mentioned above, we can find the Lorentz transformations:

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{3.3}$$
$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}. \tag{3.4}$$

These calculations are omitted for brevity.

In special relativity, it is assumed that

$$y' = y, \tag{3.5}$$

$$z' = z. \tag{3.6}$$

We discuss the problems with such assumptions in Section 10.1.

3.2 Postulates for coordinate transformation using complex numbers

The following three items are premised to find Lorentz transformations of special relativity using the coordinate transformation $\overline{A}/|A|$ with complex numbers.

(1) To add or subtract time t and length x, each unit has to be made identical. For this purpose, time is calculated in ct, which is obtained by multiplying time t by the velocity of light c. Because the unit of velocity is obtained by dividing length by time, the unit of the velocity of light becomes length, if velocity is multiplied by time.

(2) Unlike the Minkowski space–time diagram of special relativity, in which it is assumed that x is the horizontal axis and ct is the vertical axis, we assume a complex plane in which ct is the horizontal axis and x is the vertical axis. Therefore, coordinates are expressed as $(ct,\ x)$. Because the slope of a straight line expresses velocity by this method, there are advantages to drawing figures this way and calculations are easy.

(3) A complex number expressed in coordinates following the form $(ct,\ x)$ is $ct + xi$. Typically, the imaginary unit is written at the beginning of a term as ib.

However, in this book, the order is reversed so that a number is followed by the imaginary unit, i.e., bi. This practice serves the customary formalism enhances comprehension.

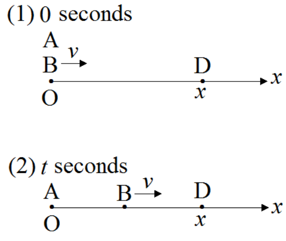


Figure 3.1

On the basis of these three preconditions, let us consider motion in the x-axis. As shown in Figure 3.1, observer A is at rest at origin O and observer B moves with a uniform velocity v in the positive x-direction. In addition, observed point D is at rest at a position of distance x and B coincides with A at origin O at time $t = 0$. After t seconds, A is at time t at distance 0; B is at time t at distance vt; and D is at time t at distance x. The coordinates are $A(ct,\ 0)$, $B(ct,\ vt)$, and $D(ct,\ x)$ and the complex numbers are $A = ct$, $B = ct + vti$, and $D = ct + xi$. Figure 3.2 depicts this representation if it is drawn in the complex plane. From now on, we call the imaginary axis the ct-axis and the real axis the xi-axis.

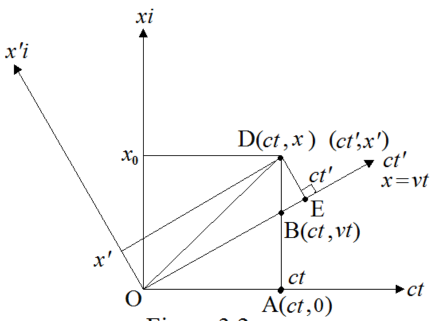


Figure 3.2

Observer A moves in the ct-axis because time t passes but distance x remains 0. The equation of the straight line is $x = 0$. Because the slope of the straight line of observer B is $vt/(ct) = v/c$, the equation of the straight line is

$$x = \frac{v}{c}(ct) = vt.$$

And B moves in the straight line $x = vt$. Note that though the slope of the straight line of B is v/c, the equation is not

$$x = \frac{v}{c}t.$$

Because the unit of the ct-axis is ct, the correct equation is as follows:

$$x = \frac{v}{c}(ct) = vt.$$

Stationary point D moves in a parallel straight line on the ct-axis because x does not change; only ct changes. Distance x of D is temporarily transformed to x_0 because it is a fixed number. And the equation of the straight line in which D moves is $x = x_0$. In addition, as seen from A at rest, the times of A, B, and D are t. Thus, A, B, and D form one straight line, which is parallel to the xi-axis. A point moving in four-dimensional space–time is called a world point and the line which the point traces out is called a world line or four-dimensional line in relativity theory. Thus, $x = 0$, $x = vt$, and $x = x_0$ are the equations of world lines of A, B, and D, respectively. In addition, the complex plane is equivalent to four-dimensional space–time, in which $y = z = 0$.

3.3 Derivation of Lorentz transformations using complex numbers

Using $\overline{A}/|A|$ for the coordinate transformation by complex numbers, $D\overline{B}/|B|$ means the coordinate transformation of D by B. In other words, $D\overline{B}/|B|$ represents the coordinates of D described by B in uniform motion.

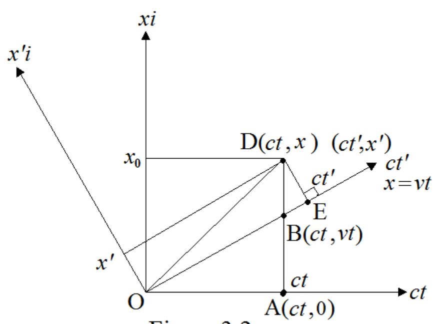


Figure 3.2

As shown in Figure 3.2, if we assume that the straight line $x = vt$ of B is a new real axis, i.e., ct'-axis, and that the straight line that meets $x = vt$ at a right angle through origin O is the new imaginary axis, i.e., the $x'i$-axis, the coordinates of D as seen from B are $(ct',\ x')$. However, we have $ct' = x' = 0$ if $ct = x = 0$ because B coincides with A at origin O at time $t = 0$. If we assume that E is the point of intersection of the ct'-axis and the perpendicular line that is drawn to $x = vt$ from D, we can write

$$\begin{aligned}\frac{D\overline{B}}{|B|} &= |OE| + |DE|\, i \\ &= ct' + x'i. \end{aligned} \tag{3.7}$$

However, $|OE|$ and $|DE|$ express distances between two points and they are not magnitudes of complex numbers, as explained before.

If $D\overline{B}/\,|B|$ is calculated using the complex numbers $B = ct + vti$ and $D = ct + xi$, the equations are

$$\begin{aligned}\frac{D\overline{B}}{|B|} &= \frac{D\overline{B}}{\sqrt{|B|^2}} \\ &= \frac{(ct + xi)(ct - vti)}{\sqrt{(ct + vti)(ct - vti)}} \\ &= \frac{c^2t^2 - cvt^2i + xcti - xvti^2}{\sqrt{c^2t^2 - v^2t^2i^2}} \\ &= \frac{c^2t^2 + xvt - cvt^2i + xcti}{\sqrt{c^2t^2 + v^2t^2}}.\end{aligned}$$

Under the condition that $ct > 0$, we have

$$\begin{aligned}\frac{D\overline{B}}{|B|} &= \frac{c^2t(t + vx/c^2) + ct(x - vt)i}{ct\sqrt{1 + v^2/c^2}} \\ &= \frac{c(t + vx/c^2) + (x - vt)i}{\sqrt{1 + v^2/c^2}}.\end{aligned} \tag{3.8}$$

Because (3.7) and (3.8) are both equal representations of $D\overline{B}/\,|B|$, we can write

$$ct' + x'i = \frac{c(t + vx/c^2) + (x - vt)i}{\sqrt{1 + v^2/c^2}}.$$

If we compare the coefficients of the imaginary parts and the real parts, we have

$$t' = \frac{t + (v/c^2)x}{\sqrt{1 + v^2/c^2}}, \tag{3.9}$$

$$x' = \frac{x - vt}{\sqrt{1 + v^2/c^2}}. \tag{3.10}$$

Equations (3.9) and (3.10) express the relations of coordinates $(ct,\ x)$ of D observed from A at rest and coordinates $(ct',\ x')$ of D observed from B with uniform velocity v. In other words, they are the equations of coordinate transformations.

Next, we compare (3.9) and (3.10) with Lorentz transformations of special relativity. As explained in Section 3.1, Lorentz transformations are

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{3.3}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}. \tag{3.4}$$

By comparing (3.3) and (3.4) with (3.9) and (3.10), we can understand that (3.9) and (3.10) will become identical with (3.3) and (3.4) if the sign of c^2 is changed into $(-)$ from $(+)$ in (3.9) and (3.10). The complex plane used to derive (3.9) and (3.10) is a flat plane. However, according to general relativity, space–time curves. In other words, (3.9) and (3.10) are Lorentz transformations in flat space–time and (3.3) and (3.4) are those in curved space–time. Differences occur depending on whether the sign of c^2 is $(+)$ or $(-)$. In addition, though mass bends space in general relativity, we can tell by our previous results that space–time is already curved in special relativity, in which mass does not yet exist.

3.4 The fourth imaginary number *h*

Hamilton expanded a complex number $a + bi$ into a quaternion $a + bi + cj + dk$ and expressed the position of a point in four-dimensional space–time. Variables $a,\ b,\ c$, and d are real numbers and $i,\ j$, and k are imaginary numbers. The algorithms are as follows:

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

Hamilton and his successor tried to describe physical laws using the quaternion in the latter half of the 19th century, but they did not succeed. The reason for their failure is thought to be that the four-dimensional space–time, where Hamilton's quaternion can be used, is flat space–time. If we intend to use the quaternion for coordinate transformations in curved space–time, an introduction of a new imaginary number is necessary. Therefore, we insert a fourth imaginary number h. The algorithms are as follows:

$$h^2 = -1,$$

$$hi = ih,\ hj = jh,\ hk = kh.$$

The algorithms of i, j, and k are identical to Hamilton's algorithms.

If we assume $hi = -ih$, calculations become complicated and we cannot find Lorentz transformations later. One can see this oneself. In addition, though i, j, and k have mutual relations as $ij = k$, h has a single relation with each imaginary number as $hi = ih$. The reason is that though we can move from space to space, we cannot move from space to time. Because h expresses time and i, j, k express space, h is unrelated to i, j, and k. In other words, we cannot mathematically move from space to time because $hi = ih$, $hj = jh$, and $hk = kh$. Time travel is not possible.

From now on, the position of a point in four-dimensional space–time is described by a new quaternion $ah + bi + cj + dk$. However, c does not represent the velocity of light any more. In addition, a is the time part and b, c, and d are the space parts. The number that is made by assuming $c = d = 0$ in the new quaternion is the new complex number $ah + bi$. ct is multiplied by imaginary number h and is calculated in cth if the new complex number is used. However, c in this case is the velocity of light. We expect that the signs of c^2 of (3.9) and (3.10) become $(-)$ from $(+)$ if we perform coordinate transformation $D\overline{B}/|B|$ using this new time unit cth because $(cth)^2 = -c^2t^2$.

3.5 Derivation of Lorentz transformations using new complex numbers

The horizontal axis becomes the cth-axis and the vertical axis becomes the xi-axis in the new complex plane if the new complex number $ah+bi$ is used. As for observer A, observer B, and stationary point D in the new complex plane, the coordinates $A(ct,\ 0)$, $B(ct,\ vt)$, and $D(ct,\ x)$ do not change; however, the new complex numbers to express their positions become $A = cth$, $B = cth + vti$, and $D = cth + xi$. If we calculate $D\overline{B}/|B|$ again, we have

$$\begin{aligned}\frac{D\overline{B}}{|B|} &= \frac{D\overline{B}}{\sqrt{|B|^2}}\\ &= \frac{(cth + xi)(cth - vti)}{\sqrt{(cth + vti)(cth - vti)}}\\ &= \frac{c^2t^2h^2 - cvt^2hi + xcthi - xvti^2}{\sqrt{c^2t^2h^2 - v^2t^2i^2}}\\ &= \frac{-c^2t^2 + xvt - cvt^2hi + xcthi}{\sqrt{c^2t^2h^2 - v^2t^2i^2}}. \qquad (3.11)\end{aligned}$$

The denominator of (3.11) is

$$\sqrt{c^2t^2h^2 - v^2t^2i^2} = \sqrt{-c^2t^2 + v^2t^2}$$
$$= \sqrt{-c^2t^2(1 - v^2/c^2)}, \tag{3.12}$$

and the sign inside the square root symbol becomes negative in the condition where $v < c$, which means that the velocity v of observer B is under the velocity of light c. Therefore, if we form an axiom: place the negative sign outside of the square root when the quantity inside the square root is negative, we find

$$\sqrt{c^2t^2h^2 - v^2t^2i^2} = cth\sqrt{1 - (v^2t^2i^2)/(c^2t^2h^2)}$$
$$= cth\sqrt{1 - v^2/c^2}, \tag{3.13}$$

because $c^2t^2h^2 = -c^2t^2 < 0$ and $ct > 0$. In addition, $\sqrt{h^2} = h$ is an axiom. An axiom is a law that is thought to be right, but cannot be proved. The two previous axioms are termed Axiom 9 and Axiom 10 in Section 13.2.

In addition, if the velocity v is beyond the velocity of light c, i.e., $v > c$, (3.12) becomes

$$\sqrt{c^2t^2h^2 - v^2t^2i^2} = \sqrt{-c^2t^2 + v^2t^2}$$
$$= \sqrt{c^2t^2(v^2/c^2 - 1)}$$
$$= ct\sqrt{v^2/c^2 - 1}.$$

Therefore, the equations of Lorentz transformations with the superluminal velocity can be obtained if coordinate transformations are performed using this denominator. We will calculate this in Chapter 19.

Then, if we apply (3.13) to (3.11) and calculate again, the equations become

$$\frac{D\overline{B}}{|B|} = \frac{D\overline{B}}{\sqrt{|B|^2}}$$
$$= \frac{-c^2t^2 + xvt - cvt^2hi + xcthi}{\sqrt{c^2t^2h^2 - v^2t^2i^2}}$$
$$= \frac{-c^2t(t - vx/c^2) + ct(x - vt)hi}{cth\sqrt{1 - v^2/c^2}}$$
$$= \frac{-(c/h)(t - vx/c^2) + (x - vt)i}{\sqrt{1 - v^2/c^2}}$$
$$= \frac{-(ch/h^2)(t - vx/c^2) + (x - vt)i}{\sqrt{1 - v^2/c^2}}$$
$$= \frac{c(t - vx/c^2)h + (x - vt)i}{\sqrt{1 - v^2/c^2}}. \tag{3.14}$$

Because the new complex number used to express the position of D as seen from B is $ct'h + x'i$, if we use the new complex number, we can write it using (3.14) as

$$ct'h + x'i = \frac{c(t - vx/c^2)h + (x - vt)i}{\sqrt{1 - v^2/c^2}}.$$

If we compare the coefficients of the imaginary parts and the real parts, we have

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{3.15}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}. \tag{3.16}$$

Equations (3.15) and (3.16) are identical to Lorentz transformations, i.e.,

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{3.3}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \tag{3.4}$$

which was explained in Section 3.1.

We can obtain Lorentz transformations after performing coordinate transformation $D\overline{B}/|B|$ using the new complex number $cth + xi$, as shown above. In other words, the fourth imaginary number h is a number necessary to describe the curved four-dimensional space–time and is not a number of suppositions. From now, we express the position of a point in four-dimensional space–time with the new quaternion $cth + xi + yj + zk$ and the position in the new complex plane by the condition $y = z = 0$.

In special relativity, there is a case in which time ct is multiplied by imaginary number i and calculated as cti. However, in that case, lengths x, y, and z are calculated as real numbers. Therefore, cti is a number unlike cth of the new quaternion. Because cti in relativity theory is used for calculation convenience, cti does not express the actual condition of space–time. On the contrary, cth of the new quaternion $cth + xi + yj + zk$ expresses the property of curved space–time and cth is the number that is necessary for investigations of space–time.

In addition, though we use the condition $ct > 0$ to obtain (3.14), we do not make it necessary that $t > 0$. We find $ct > 0$ if $c < 0$ and $t < 0$. Therefore, we can obtain Lorentz transformations under the condition of $t < 0$. However, because the universality of Lorentz transformations is lost if the sign of the velocity of light c changes in the case of positive or negative time t, we will continue under the condition that c is always positive.

3.6 The reason why time advances only in the positive direction

In the last section, we assumed that the velocity of light c is always positive. If that assumption is used, we can explain the reason why time advances only in the positive direction in our world.

If we had calculated:

$$\frac{D\overline{B}}{|B|} = \frac{(cth + xi)(cth - vti)}{\sqrt{(cth + vti)(cth - vti)}} = \frac{-c^2t^2 + xvt - cvt^2hi + xcthi}{\sqrt{c^2t^2h^2 - v^2t^2i^2}} \tag{3.11}$$

in the last section under the conditions of $c > 0,\ t < 0$, i.e., $ct < 0$, we would have

$$\begin{aligned}\frac{D\overline{B}}{|B|} &= \frac{-c^2t^2 + xvt - cvt^2hi + xcthi}{\sqrt{c^2t^2h^2 - v^2t^2i^2}} \\ &= \frac{-c^2t(t - vx/c^2) + ct(x - vt)hi}{-cth\sqrt{1 - v^2/c^2}} \\ &= \frac{(c/h)(t - vx/c^2) - (x - vt)i}{\sqrt{1 - v^2/c^2}} \\ &= \frac{c(vx/c^2 - t)h + (vt - x)i}{\sqrt{1 - v^2/c^2}}.\end{aligned} \tag{3.17}$$

Because the new complex number expressing the position of D as seen from B is $ct'h + x'i$, we can write it using equation (3.17) as

$$ct'h + x'i = \frac{c(vx/c^2 - t)h + (vt - x)i}{\sqrt{1 - v^2/c^2}}.$$

If we compare coefficients of the imaginary parts and real parts, we have

$$t' = \frac{(v/c^2)x - t}{\sqrt{1 - v^2/c^2}}, \tag{3.18}$$

$$x' = \frac{vt - x}{\sqrt{1 - v^2/c^2}}.$$

The axiom $\sqrt{h^2} = h$ is used here.

In the case of $x = 0$ in (3.18), we have

$$t' = \frac{-t}{\sqrt{1 - v^2/c^2}}.$$

From this equation, the sign of time t of D as seen from A is reversed compared to the sign of time t' of D as seen from B. Because B coincides with A if the velocity of B is 0, it is unnatural that the sign of t should be reversed with respect to t'. Thus, the first suppositions of $c > 0,\ t < 0$ are thought to be incorrect. In other words, according to the calculations, time t can move only in the positive direction in our world. This is the reason why we cannot return to the past.

In addition, we will discuss whether we can go against time in the superluminal velocity region in Section 19.2.

4

New Complex Plane and Oblique Coordinate Axes

4.1 New complex number to express the rotation of coordinate axes

Because the complex plane is a plane without a curve, an argument does not change by rotation. Moreove, if we assume that a straight line $x = vt$ is the new real axis, i.e., the ct'-axis, as shown in Figure 3.2, the new imaginary axis, i.e., the $x'i$-axis, is a straight line that is perpendicular to the straight line $x = vt$ through origin O.

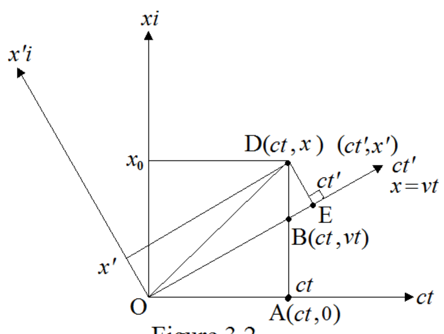


Figure 3.2

However, because in reality the new complex plane is a plane that curves, it is unknown where the new imaginary axis, i.e., the $x'i$-axis, moves to. It is difficult to use a figure if we do not know the placement of the $x'i$-axis. In this section, we calculate the place where the $x'i$-axis moves to. This calculation is necessary to examine whether the new imaginary number h is a fictitious number or an actual number.

If observer B moves in the x-direction with respect to observer A with uniform velocity v, the new complex numbers A and B are $A = cth$ and $B = cth + vti$, respectively. We considere a case, in which the time of A is t_0. Because the time t_0 of A and B are the same, we have Figure 4.1.

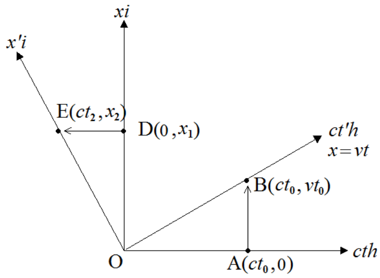


Figure 4.1

At this time, by coordinate transformations, point $A(ct_0,\ 0)$ in the cth-axis moves to point $B(ct_0,\ vt_0)$ in the $ct'h$-axis and point $D(0,\ x_1)$ in the xi-axis moves to point $E(ct_2,\ x_2)$ in the $x'i$-axis. However, all the coordinates of $A,\ B,\ D$, and E are the coordinates seen from A. If the coordinates of B are the coordinates seen from B, imaginary number h of new complex number $A = ct_0h$ and h of new complex number $B = ct_0h + vt_0i$ are not in the same axis. In that case, the following discussions are not realized.

Because the movement from A to B is a rotation around origin O, it can be expressed by new complex number H. In addition, if it is not a rotation around origin O, the movement is not expressed by a new complex number.Because the new complex numbers of A and B are $A = ct_0h$ and $B = ct_0h + vt_0i$, respectively, using $AH = B$, we have

$$\begin{aligned} ct_0hH &= ct_0h + vt_0i, \\ H &= \frac{ct_0h + vt_0i}{ct_0h} \qquad (\because\ t_0 \neq 0) \\ &= 1 + \frac{vi}{ch} \\ &= 1 + \frac{vih}{ch^2} \\ &= 1 - \frac{v}{c}hi. \end{aligned} \tag{4.1}$$

(4.1) changes the cth-axis into the $ct'h$-axis of the new complex number system.

4.2 Proof of the oblique coordinate axes

New complex numbers expressing the positions of D and E in Figure 4.1 are $D = x_1i$ and $E = ct_2h + x_2i$, respectively. Because D is moved to E by transformation H, which moves A to B, using $DH = E$ we have

$$x_1 iH = ct_2 h + x_2 i.$$

If (4.1) is substituted for this equation, we find

$$\begin{aligned} ct_2 h + x_2 i &= x_1 i(1 - \frac{v}{c} hi) \\ &= x_1 i - \frac{vx_1}{c} hi^2 \\ &= x_1 i + \frac{v}{c} x_1 h \\ &= \frac{v}{c} x_1 h + x_1 i. \end{aligned}$$

After comparing the coefficients, we have

$$ct_2 = \frac{v}{c} x_1, \;\; x_2 = x_1.$$

And if x_1 is eliminated from the two formulae, we can write

$$x_2 = \frac{c}{v}(ct_2). \tag{4.2}$$

Equation (4.2) is an equation of the $x'i$-axis. Because the unit of the temporal axis is ct, the slope of the $x'i$-axis is not c^2/v but c/v. The equation of the $ct'h$-axis, to which the cth-axis moves, is as follows:

$$x = vt = \frac{v}{c}(ct).$$

Thus, the slope of the $ct'h$-axis becomes v/c and as shown in Figure 4.2, the $x'i$- and the $ct'h$-axes lean inward at the same arguments from the xi- and cth-axes. In other words, the $x'i$- and $ct'h$-axes are oblique coordinate axes.

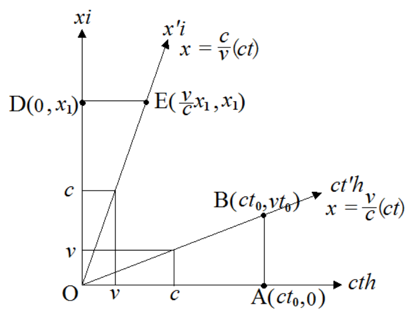


Figure 4.2

From the above results, if we move one axis of the coordinate axes that make a couple, a pair of new coordinate axes become oblique coordinate axes in the new complex plane. Although coordinate axes become oblique coordinate axes in special relativity as well, a time part is calculated by temporarily turning it into imaginary number cti because the axes become rectangular coordinate axes through calculations using ct. However, as stated above, the special operation for making oblique coordinate axes is not necessary with the new complex numbers.

Because the Minkowski space–time diagram is a flat surface, if the coordinate axes rotate, they become rectangular coordinate axes. On the other hand, in the new complex plane, the rotations of coordinate axes make oblique coordinate axes. The new complex plane improves upon the Minkowski space–time diagram in this regard as well.

4.3 A simple method to obtain oblique coordinate axes

We have come to understand that the coordinate axes in the new complex plane are oblique coordinate axes. Now we consider the method by which we can obtain the equation of a coordinate system that makes a pair when the equation of another coordinate axis is given.

The equation of the world line of light is $x = ct$, which means that light is a straight line equidistant from both coordinate axes. Because oblique coordinate axes inwardly declining from the orthogonal coordinate axes are linearly symmetric, we have the same result even if we replace the cth- and xi-axes. Therefore, we can have the equations of the coordinate axes that make a pair if we switch x and ct in a given equation. For example, if we switch ct and x in the equation

$$x = vt = \frac{v}{c}(ct)$$

of the $ct'h$-axis in Figure 4.2, we have

$$ct = \frac{v}{c}x.$$

If the equation is transformed, it becomes

$$x = \frac{c}{v}(ct),$$

and it corresponds with (4.2).

If we use this method, we can easily have the equations of coordinate axes that make a pair if the coordinate axes are curved. Curved coordinate axes mean that observer B is accelerating. We argue this point in Section 20.2.

5

Length in the New Complex Plane and Lorentz Transformations

5.1 Method of obtaining Lorentz transformations from length in the new complex plane

As proven in Section 3.5, the new complex numbers of observer A at rest, observer B moving along a straight line with constant velocity v, and observed point D are $A = cth$, $B = cth+vti$, and $D = cth+xi$, respectively. The Lorentz transformations can be found by coordinate transformation $D\overline{B}/|B|$.

There exists another method to find the Lorentz transformations by the coordinates of the nodes of the world lines in the new complex plane. This is the second method to find the Lorentz transformations using the new complex number. By this method, neither emission of light, nor the coordinate transformation by the new complex number, is needed.

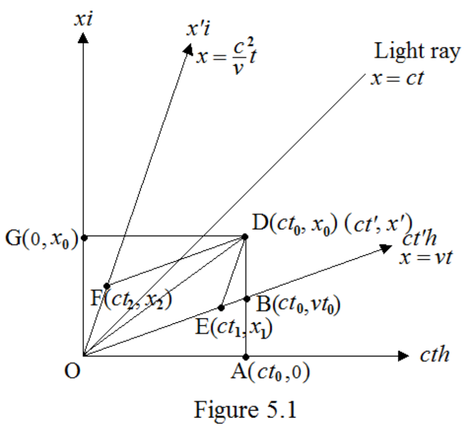


Figure 5.1

It is assumed that observer A is at rest at the origin O, and observer B moves with constant velocity v in the positive direction along the x-axis. The observed point D is at rest at position x_0. In addition, it is assumed that A coincides B at O at time $t = 0$. The coordinates of A, B, and D after t_0 seconds are $A(ct_0,\ 0)$, $B(ct_0,\ vt_0)$, and $D(ct_0,\ x_0)$, respectively. Figure 5.1 shows these coordinates drawn in a new complex plane.

A moves along the cth-axis. B moves along the straight line $x = vt$, i.e., the $ct'h$-axis. The node where the perpendicular line from D meets the xi-axis is G, and D moves along the straight line GD. The node where the straight line $x = vt$ meets the straight line drawn from D and is parallel to the $x'i$-axis, is E. The node where the $x'i$-axis meets the straight line drawn from D and is parallel to the $ct'h$-axis, is F. As explained in Section 4.3, if x and ct are switched in the equation

$$x = vt = \frac{v}{c}(ct)$$

of the $ct'h$-axis, we have

$$ct = \frac{v}{c}x.$$

The transformed equation

$$x = \frac{c^2}{v}t$$

corresponds to the $x'i$-axis.

The coordinates $(ct',\ x')$ of D, seen from the $ct'h$- and $x'i$-axes, which are the oblique coordinate axes of observer B, are the lengths of nodes E and F from O. Then, we can write

$$ct' = |OE| = |DF|\,,\ x' = |OF| = |DE|\,. \tag{5.1}$$

However, $|OE|\,,\ |DF|\,,\ |OF|$, and $|DE|$ do not express the magnitudes of the complex numbers but the lengths between the two points. As is shown below, we find t' and x' from the coordinates of the nodes of two straight lines.

(1) t' is obtained from the coordinates of point E

The slope of the $x'i$-axis is c/v, as can be seen from the equation

$$\begin{aligned} x &= \frac{c^2}{v}t \\ &= \frac{c}{v}(ct). \end{aligned}$$

Because straight line DE is parallel to the $x'i$-axis, its argument is c/v. Also, because straight line DE passes through point $D(ct_0,\ x_0)$, its equation is

$$x - x_0 = \frac{c}{v}(ct - ct_0),$$
$$x = \frac{c^2}{v}(t - t_0) + x_0. \tag{5.2}$$

In addition, the equation of straight line OB is

$$x = vt. \tag{5.3}$$

Thus, if x is eliminated from (5.2) and (5.3), we have

$$\frac{c^2}{v}(t - t_0) + x_0 = vt,$$
$$\frac{c^2}{v}t - vt = \frac{c^2}{v}t_0 - x_0,$$
$$\frac{(c^2 - v^2)}{v}t = \frac{c^2}{v}t_0 - x_0,$$
$$(c^2 - v^2)t = c^2 t_0 - vx_0,$$
$$t = \frac{c^2 t_0 - vx_0}{c^2 - v^2}.\ (\because\ c \neq v) \tag{5.4}$$

If t is eliminated from (5.3) and (5.4), we find

$$x = \frac{v(c^2 t_0 - vx_0)}{c^2 - v^2}. \tag{5.5}$$

If the coordinates of E are $(ct_1,\ x_1)$, then from (5.4) and (5.5), we have

$$ct_1 = \frac{c(c^2 t_0 - vx_0)}{c^2 - v^2},\ x_1 = \frac{v(c^2 t_0 - vx_0)}{c^2 - v^2}. \tag{5.6}$$

However, $(ct_1,\ x_1)$ are the coordinates as seen from the cth- and xi-axes.

From (5.6), as well as the definition of length by the new complex number, length $|OE|$ is obtained by the definition of magnitude. We find

$$\begin{aligned}|OE|^2 &= E\overline{E} \\ &= (ct_1 h + x_1 i)(ct_1 h - x_1 i) \\ &= -c^2 t_1^2 + x_1^2 \\ &= -\frac{c^2(c^2 t_0 - vx_0)^2}{(c^2 - v^2)^2} + \frac{v^2(c^2 t_0 - vx_0)^2}{(c^2 - v^2)^2} \\ &= -\frac{(c^2 t_0 - vx_0)^2}{c^2 - v^2} \\ &= -\frac{c^2(t_0 - vx_0/c^2)^2}{1 - v^2/c^2}.\end{aligned} \tag{5.7}$$

Since

$$ct' = |OE| = |DF|\,,\ x' = |OF| = |DE| \tag{5.1}$$

and since the new complex number of point E as seen from B after the coordinate transformation is $ct'h$, we have

$$\begin{aligned} |OE|^2 &= (ct'h)^2 \\ &= -c^2t'^2. \end{aligned} \tag{5.8}$$

Because (5.7) and (5.8) are the same, we can write

$$\begin{aligned} -c^2t'^2 &= -\frac{c^2(t_0 - vx_0/c^2)^2}{1 - v^2/c^2}, \\ t'^2 &= \frac{(t_0 - vx_0/c^2)^2}{1 - v^2/c^2}. \end{aligned} \tag{5.9}$$

To find t' from (5.9), we consider the signs of the numerator and denominator of (5.9). If $c > v$, the sign of the denominator of (5.9) is

$$1 - v^2/c^2 > 0. \tag{5.10}$$

Calculations to determine the sign of the numerator are somewhat complex.

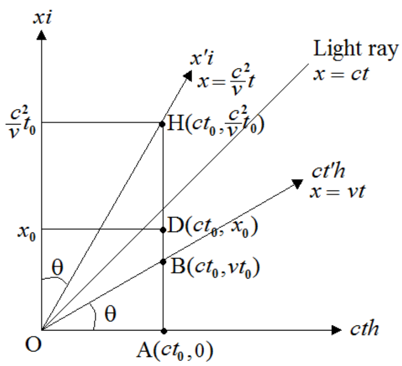


Figure 5.2

As shown in Figure 5.2, the coordinates of the node where straight line AD meets the $x'i$-axis is H. The x coordinate is obtained by substituting t for t_0 in the equation of the x'-axis, i.e.,

$$x = \frac{c^2}{v}t,$$

and we find

$$x = \frac{c^2}{v}t_0.$$

Because point $D(ct_0,\ x_0)$ lies vertically below point H, we have

$$\frac{c^2}{v}t_0 \geqq x_0.$$

Therefore, from the numerator of (5.9), we can write

$$t_0 - vx_0/c^2 \geqq t_0 - \frac{c^2}{v}t_0(v/c^2) = t_0 - \frac{c^2vt_0}{c^2v} = 0$$

or in short,

$$t_0 - vx_0/c^2 \geqq 0. \tag{5.11}$$

From (5.10) and (5.11), (5.9) becomes

$$t' = \frac{t_0 - (v/c^2)x_0}{\sqrt{1 - v^2/c^2}}.$$

If t_0 and x_0 are replaced with t and x in order to use generalized notation, this becomes

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}},$$

which is a standard equation of the Lorentz transformations, i.e.,

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}. \tag{3.3}$$

As shown above, the equation for t' in the Lorentz transformations can be found from the coordinates of the node of the world lines in the new complex plane.

(2) x' is obtained from the coordinates of point F

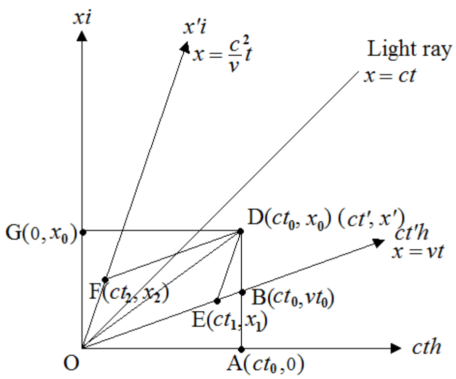


Figure 5.1

In Figure 5.1, from the equation

$$\begin{aligned} x &= vt \\ &= \frac{v}{c}(ct) \end{aligned}$$

of the $ct'h$-axis, the slope is v/c. Because straight line DF is parallel to the $ct'h$-axis, the slope is v/c. Thus, because straight line DF passes through point $D(ct_0,\ x_0)$, its equation becomes

$$\begin{aligned} x - x_0 &= \frac{v}{c}(ct - ct_0), \\ x &= v(t - t_0) + x_0. \end{aligned} \tag{5.12}$$

Because straight line OF lies along the $x'i$-axis, its equation becomes

$$x = \frac{c^2}{v}t. \tag{5.13}$$

If x is eliminated from (5.12) and (5.13), we have

$$\begin{aligned} \frac{c^2}{v}t &= v(t - t_0) + x_0, \\ \frac{(c^2 - v^2)}{v}t &= -vt_0 + x_0, \\ t &= \frac{v(x_0 - vt_0)}{c^2 - v^2}.\ (\because\ c \neq v) \end{aligned} \tag{5.14}$$

If t is eliminated from (5.13) and (5.14), we have

$$x = \frac{c^2(x_0 - vt_0)}{c^2 - v^2}. \tag{5.15}$$

If the coordinates of point F are $(ct_2,\ x_2)$, then from (5.14) and (5.15), we find

$$ct_2 = \frac{cv(x_0 - vt_0)}{c^2 - v^2},\ x_2 = \frac{c^2(x_0 - vt_0)}{c^2 - v^2}. \tag{5.16}$$

However, $(ct_2,\ x_2)$ are also the coordinates as seen from the cth- and xi-axes.

From the definition of the square magnitude by the new complex number and from (5.16), we can write

$$\begin{aligned} |OF|^2 &= F\overline{F} \\ &= (ct_2h + x_2i)(ct_2h - x_2i) \\ &= -c^2t_2^2 + x_2^2 \end{aligned}$$

$$= -\frac{c^2v^2(x_0 - vt_0)^2}{(c^2 - v^2)^2} + \frac{c^4(x_0 - vt_0)^2}{(c^2 - v^2)^2}$$
$$= \frac{c^2(x_0 - vt_0)^2}{c^2 - v^2}$$
$$= \frac{(x_0 - vt_0)^2}{1 - v^2/c^2}. \tag{5.17}$$

Because the equations

$$ct' = |OE| = |DF|\,,\ x' = |OF| = |DE| \tag{5.1}$$

show that the new complex number of point F as seen from B after the coordinate transformation is $x'i$, we have

$$|OF|^2 = (x'i)(-x'i)$$
$$= x'^2. \tag{5.18}$$

Because (5.17) and (5.18) are the same, we can write

$$x'^2 = \frac{(x_0 - vt_0)^2}{1 - v^2/c^2}. \tag{5.19}$$

If $c > v$, the sign of the denominator of (5.19) is

$$(1 - v^2/c^2) > 0. \tag{5.20}$$

Because point D lies above point B in Figure 5.2, we have $x_0 \geqq vt_0$.

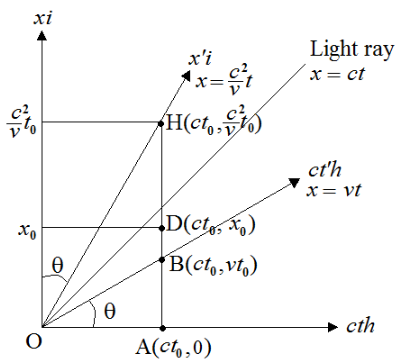


Figure 5.2

Therefore, the sign of the numerator of (5.19) becomes

$$x_0 - vt_0 \geqq 0. \tag{5.21}$$

From (5.20) and (5.21), (5.19) becomes

$$x' = \frac{x_0 - vt_0}{\sqrt{1 - v^2/c^2}}.$$

If t_0 and x_0 are replaced with t and x in order to use generalized notation, this becomes

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}},$$

which is the standard equation of the Lorentz transformations, i.e.,

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}. \tag{3.4}$$

As shown above, the equation for x' in the Lorentz transformations can be found from the coordinates of the node of the world lines in the new complex plane.

In the above verifications, the Lorentz transformations were obtained by the following methods without emission of light and without using the coordinate transformation by the new complex number.

1) The coefficient, which changes time into length, is c, and time is calculated by ct.
2) The rest frame of observer A is made into a rectangular frame and the world line of moving observer B is set as the temporal axis of the oblique frame. The equation of the length axis of the oblique frame is obtained by switching x and ct in the equation of the temporal axis in this frame.
3) The coordinates of nodes E or F of each oblique coordinate axis and the straight line, which is drawn from observed point D and parallel to the oblique coordinate axis, are calculated from the equations of the world lines.
4) If the coordinates of nodes E or F are $(ct,\ x)$, the world distance is calculated by

$$\sqrt{(cth + xi)(cth - xi)},$$

because the new complex number of that point is $cth + xi$.
5) The length from O to point E or F on the oblique coordinate axis becomes the coordinates $(ct',\ x')$ of the point at rest in the oblique frame.

This result shows that the Lorentz transformations can be obtained using only the new complex number and the new complex plane. In contrast, in Minkowski spacetime diagrams, Lorentz transformations cannot be obtained from the coordinates of the nodes of world lines.

5.2 Equation of a straight line through new complex numbers

Though the new complex plane is a plane consisting of two imaginary axes, why can we obtain the nodes using only the equations of real numbers? To answer this, we investigate whether the equation of the straight line in the new complex plane may be written with real numbers.

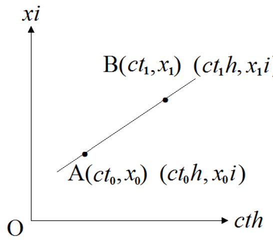


Figure 5.3

As shown in Figure 5.3, there are two points A and B in the new complex plane having coordinates $(ct_0,\ x_0)$ and $(ct_1,\ x_1)$, respectively. If the equation corresponding to the straight line that passes through them is calculated with real numbers, the slope is

$$\frac{x_1 - x_0}{ct_1 - ct_0}$$

and it passes through point $A(ct_0,\ x_0)$; hence, the equation of the straight line is

$$x - x_0 = \frac{x_1 - x_0}{ct_1 - ct_0}(ct - ct_0).$$

The equation becomes

$$x = \frac{x_1 - x_0}{t_1 - t_0}(t - t_0) + x_0. \tag{5.22}$$

Then, the equation corresponding to the straight line is calculated by the new complex number. The points A and B are displayed as $A(ct_0h,\ x_0i)$ and $B(ct_1h,\ x_1i)$, respectively. Because the slope is

$$\frac{x_1i - x_0i}{ct_1h - ct_0h}$$

and it passes through point $A(ct_0h,\ x_0i)$, the equation of the straight line is

$$xi - x_0i = \frac{x_1i - x_0i}{ct_1h - ct_0h}(cth - ct_0h).$$

These equations become

$$(x - x_0)i = \frac{(x_1 - x_0)i}{(t_1 - t_0)ch}(t - t_0)ch,$$

$$x - x_0 = \frac{x_1 - x_0}{t_1 - t_0}(t - t_0),$$
$$x = \frac{x_1 - x_0}{t_1 - t_0}(t - t_0) + x_0. \tag{5.23}$$

(5.23) is the same as (5.22). In other words, even if the equation of the straight line is calculated by new complex numbers, it will be the same equation as that obtained by real numbers. Thus, the straight line in the new complex plane can be calculated by equations of real numbers.

5.3 Method to express coordinate parts through new complex numbers

Hamilton invented the method of denoting coordinate parts by the combination of real numbers $(a,\ b)$ if the complex number $a + bi$ is drawn in the complex plane. If we see the coordinate axes in the complex plane, we can easily determine whether a and b express real or imaginary parts. However, Hamilton's quaternion consists of one real number and three imaginary numbers, and the new quaternion consists of four imaginary numbers. Thus, we must determine whether the four real numbers $(ct,\ x,\ y,\ z)$ are real or imaginary by placing them on the coordinate axes. To solve this inconvenience, real and imaginary numbers are written in (, , ,) form from now on. We can see that no contradictions occur if we change into this notation, because the coordinates of point $A(ct_0,\ x_0)$ can be calculated by writing them as $A(ct_0h,\ x_0i)$, as explained in the last section.

The reader who begins with Chapter 6 without first reading Chapters 1-5 may think that the notation of writing imaginary numbers in (,) is wrong. However, from Chapter 6, imaginary numbers are written in this way because of the reason explained above. Since the meaning can be easily understood if imaginary numbers are written in (,), we could use the new notation from Chapter 1. However, because the explanation of the new complex number was not completed, Hamilton's notation was used until Chapter 5.

5.4 Correctness of the imaginary number *h*

As explained in Section 3.4, in order for the equations of the coordinate transformations by the complex number and the equations of Lorentz transformations of special relativity to coincide, a fourth imaginary number h was introduced, in addition to Hamilton's three imaginary numbers $i,\ j$, and k. Because calculations in curved space–time can be performed without contradictions by h, it is appropriate to think of h as having the ability to express the property of this curved four-dimensional space–time.

Typical mathematics and physics explain the phenomenon in flat space–time. Though tensors are used to apply the mathematics and physics to curved space-time, it is sometimes difficult to master the tensor since the symbols are complex. However, as shown in Section 3.5 and Section 5.1, if the time part is multiplied by h and length parts by i, j, and k, the mathematics and physics required for curved space–time can be derived by ordinary methods. This is a result of the fact that the new quaternion itself contains curved properties. Curved space–time becomes flat when we apply the new quaternion. Thus, we could precisely explain the physical phenomenon in four-dimensional space–time using the new quaternion.

6

Time Dilation and Length Contraction

6.1 Proper time and time dilation

Proper time τ (tau) is an important concept next to Lorentz transformations in special relativity. Most of the conclusions of special relativity can be obtained using Lorentz transformations and the proper time. Proper time is the time measured by the inertial frame. The time of observer B and observed point D is proper time if B comes next to and coincides with D. Because it is difficult to express the proper time in this explanation, the Minkowski space-time diagram is used to explain it visually. However, because the slope of the world line is the inverse of the velocity in the Minkowski space-time diagram, it is difficult to understand proper time. In addition, in the diagram, the length of the world line is incalculable from the coordinates of the nodes of world lines. However, because the slope of the world line expresses the velocity, and the length can be calculated from the coordinates of the node in the new complex plane, we explain the proper time using the new complex plane in this section.

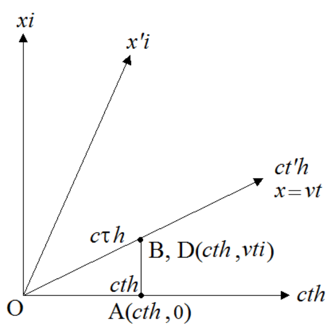


Figure 6.1

Although observed point mass D was at rest prior to now, we currently assume that D is in a state of uniform linear motion, similar to observer B. Also, as shown in Figure 6.1, it is assumed that observer B coincides with D. At this time, the velocity of D becomes v, and the straight line $x = vt$ becomes the temporal axis, i.e., the $ct'h$-axis, of D. In other words, the $ct'h$-axis is the world line of B and D. As seen from B and D, their length x' is always 0, but time t' passes. Because $|OD|$ is the length from origin O to point D in the $ct'h$ axis in Figure 6.1, $|OD|$ is the proper time τ of D. In other words,

$$|OD| = c\tau h.$$

The reason why we cannot remove h, which is attached to $|OD|$ in this equation, is explained as follows. Using the new complex conjugate $\overline{D} = c\tau h$ and the definition of magnitude, i.e., $|A| = A\overline{A}$, we have

$$\begin{aligned} |OD|^2 &= D\overline{D} \\ &= (c\tau h)^2 \\ &= -c^2\tau^2. \end{aligned} \tag{6.1}$$

Because $-c^2\tau^2 < 0$, using the axiom that place the negative sign outside of the square root when the quantity inside the square root is negative, we can write

$$\begin{aligned} |OD| &= \sqrt{-c^2\tau^2} \\ &= c\tau h. \end{aligned}$$

As this result shows, in curved space–time, the magnitude of time is an imaginary number, and thus, its square is negative. In flat space–time, the magnitude $|A|$ is a positive real number, whereas $|A|$ is a positive real number or a positive imaginary number in curved space-time.

Then, we obtain the relations between the proper time τ of moving point mass D and the time t of observer A at rest. Because the coordinates of D as seen from A are $(cth,\ vti)$ in Figure 6.1 and the new complex number of D is $D = cth + vti$, we have

$$\begin{aligned} |OD|^2 &= D\overline{D} \\ &= (cth + vti)(cth - vti) \\ &= (cth)^2 - (vti)^2 \\ &= -c^2t^2 + v^2t^2 \\ &= -c^2t^2(1 - v^2/c^2). \end{aligned} \tag{6.2}$$

From (6.1) and (6.2), we have

$$-c^2\tau^2 = -c^2t^2(1 - v^2/c^2),$$
$$\tau^2 = t^2(1 - v^2/c^2).$$

If $c > v,\ t > 0$, and $\tau > 0$, we find

$$\tau = t\sqrt{1 - v^2/c^2}. \tag{6.3}$$

In (6.3), we have

$$1 > \sqrt{1 - v^2/c^2}.$$

Thus, we find $\tau < t$, and it has therefore been shown that the proper time τ of moving point mass D is less than the time t of observer A at rest. This is the famouse time-dilation theory for a moving clock in special relativity. In addition, because time is an imaginary quantity in our world as previously explained, it is correct to write (6.3) as

$$\tau h = th\sqrt{1 - v^2/c^2}.$$

However, it is usually written in the form of (6.3), because only a real part can be observed.

In Figure 6.1, it appears that $|OD| > |OA|$. In other words, we can see that $\tau > t$. However, $\tau < t$ in (6.3); thus, there appears to be a contradiction. However, because the new complex plane represents curved two-dimensional space–time, a line segment parallel to a coordinate axis is the longest, and it becomes shorter when inclined. The straight line, which is maximally inclined most from both the cth- and xi-axes, is a light ray, and the length of the ray from the origin is 0. These results can be confirmed by calculating the magnitudes $|OP|$, $|OQ|$, and $|OR|$ after arranging the points $P(ch,\ 0)$, $Q(ch,\ i)$, and $R(ch,\ ci)$ in the new complex plane. These calculations are not presented here.

Many researchers in the field of special relativity think that the proper time τ of the point mass is invariant and they transform (6.3) and express the time t of observer A by

$$t = \frac{\tau}{\sqrt{1 - v^2/c^2}}. \tag{6.4}$$

If v changes, t changes in (6.4). However, τ is invariant. Although moving point mass D is only one example, many observers exist with its velocity. In other words, because the invariant time is τ only, it is called the proper time of the point mass. From now on, we consider physical phenomena with respect to the moving point mass. As measured by an observer co-moving with the point mass, the length is called proper length, and the mass is called relativistic mass or rest mass.

The trace of a point mass moving in the four-dimensional space–time is called the world line in relativity theory, and the length of the world line is called the world distance or four-dimensional distance. Because $|OD|$ is the world distance s of point mass D, we can write $s = c\tau h$. In Section 7.1, it will be explained that the world distance is invariant in Lorentz transformations in special relativity. Thus, the proper time is also invariant.

6.2 Special relativity and length contraction

The length of a rod moving with velocity v shortens in special relativity. The proof in special relativity is explained here before being proven with the method using the new complex number. This is the proof written in the book of relativity theory.

As shown in Figure 6.2, a rod D with length l moves with uniform, linear motion along the x-axis. It is assumed that the two-dimensional space–time coordinates of the ends of the rod as seen from observer A at rest are $(ct_1,\ x_1)$ and $(ct_2,\ x_2)$, and those from observer B moving with uniform velocity v are $(ct'_1,\ x'_1)$ and $(ct'_2,\ x'_2)$.

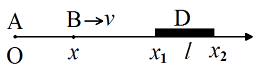


Figure 6.2

From the Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \tag{3.4}$$

we can write

$$x'_1 = \frac{x_1 - vt_1}{\sqrt{1 - v^2/c^2}}, \tag{6.5}$$

$$x'_2 = \frac{x_2 - vt_2}{\sqrt{1 - v^2/c^2}}. \tag{6.6}$$

If (6.5) is subtracted from (6.6), we find

$$x'_2 - x'_1 = \frac{(x_2 - x_1) - v(t_2 - t_1)}{\sqrt{1 - v^2/c^2}}. \tag{6.7}$$

Because the times of the both ends of the rod are the same as seen from observer A, we can write $t_1 = t_2$. Therefore, (6.7) becomes

$$x'_2 - x'_1 = \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}}. \tag{6.8}$$

If the velocity of B coincides with D, the length of the rod measured by B is called the proper length. Thus, $x'_2 - x'_1$ is the proper length l_0. In addition, the length $x_2 - x_1$ of the rod measured by A is the length l of the rod that changes with different observers. Thus, from (6.8), we have

$$l_0 = \frac{l}{\sqrt{1 - v^2/c^2}}.$$

The equation becomes

$$l = l_0\sqrt{1 - v^2/c^2}. \tag{6.9}$$

Since $\sqrt{1 - v^2/c^2} < 1$, (6.9) shows that $l < l_0$. In other words, if a rod with length l_0 is at rest at A and has velocity v after leaving from A, the length l shortens if it is measured by A. In addition, the proper length l_0 measured by B, who is moving with rod D, does not change.

6.3 Coordinates of the node of the world lines and length contraction

In this section, (6.9) is found by using the length of the world line to show that the conclusion of special relativity is obtained from the coordinates of the node of the world lines. For that purpose, it is considered to be sufficient to apply the same method used to calculate the proper time. However, the method to obtain the proper length is somewhat complex. Figure 6.3 shows the new complex plane.

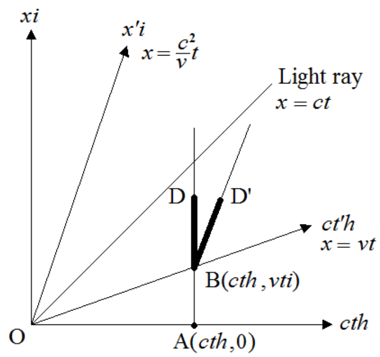


Figure 6.3

We assume that the position of observer B coincides with the near edge of the rod, and the tip of the rod is point D. Because B coincides with the rod, the velocities of B and the rod are both v. The straight line $x = vt$ becomes the temporal axis, i.e., the $ct'h$-axis, of B and the rod. The straight line

$$x = \frac{c^2}{v}t$$

that is the linear symmetry for a light ray of the straight line $x = vt$ becomes the $x'i$-axis. The equation of the straight line of linear symmetry for a light ray is obtained by switching x and ct in the equation $x = vt$ as explained in Section 4.3.

The slope of the rod is the problem in Figure 6.3. If the times of both ends of the rod measured by B are t', the rod is parallel to the $x'i$ axis in the new complex plane. In other words, the tip D' of the rod is inclined in the new complex plane. In this case, because the times of each end of the rod measured by observer A are different, this observer cannot measure the length of the rod. Therefore, because the times of each end of the rod must be the same time t as seen from A, tip D of the rod and the near edge of the rod, i.e., B , lies on the same line of the same time t in the new complex plane. In other words, the rod becomes perpendicular. An important point is that the times of D and B, i.e., the ends of the rod, as seen from observer A at rest must be the same.

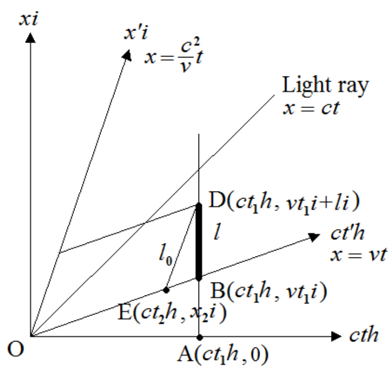


Figure 6.4

In Figure 6.4, E is the node where the $ct'h$-axis meets the straight line, which is parallel to the $x'i$-axis, and drawn from point D. In this case, $|DE|$ is the proper length l_0 of the rod as seen from B. This is due to the fact that the times of D and E are the same as seen from B. In addition, $|BD|$ is the length of the rod l as seen from A. Point B is on $x = vt$, and the coordinates of B are $(ct_1h,\ vt_1i)$ if the coordinates $(ct_1h,\ 0)$ of point A are assumed. Furthermore, because point D is at a distance l higher than point B in the xi-direction, the coordinates of D are $(ct_1h,\ vt_1i + li)$. Since straight line DE is parallel to the $x'i$-axis, i.e., straight line

$x = (c/v)ct$, and since it passes through the point $D(ct_1 h,\ vt_1 i + li)$, the equation of DE becomes

$$x - (vt_1 + l) = \frac{c}{v}(ct - ct_1). \tag{6.10}$$

The real part $(ct_1,\ vt_1 + l)$ of point $D(ct_1 h,\ vt_1 i + li)$ is used here, because even if new complex numbers or real numbers are used, the equations are the same as those presented in Section 5.2.

Because the coordinates of point E are the coordinates of the point of the node where straight line DE meets the straight line $x = vt$, we can eliminate x from (6.10) and $x = vt$, leading to

$$\begin{aligned} vt - (vt_1 + l) &= \frac{c}{v}(ct - ct_1), \\ (c^2/v - v)t &= (c^2/v - v)t_1 - l, \\ t &= t_1 - l/(c^2/v - v) \\ &= t_1 - lv/(c^2 - v^2). \end{aligned} \tag{6.11}$$

By eliminating t from (6.11) and $x = vt$, we find

$$x = vt_1 - lv^2/(c^2 - v^2). \tag{6.12}$$

If we assume that $(ct_2 h,\ x_2 i)$ are the coordinates of point E, from (6.11) and (6.12), we can write

$$ct_2 = ct_1 - clv/(c^2 - v^2), \tag{6.13}$$

$$x_2 = vt_1 - lv^2/(c^2 - v^2). \tag{6.14}$$

If the new complex number expressing DE is written as (DE), from $D(ct_1 h,\ vt_1 i + li)$, (6.13), and (6.14), the equations become

$$\begin{aligned} (DE) &= D - E \\ &= [ct_1 h + (vt_1 + l)i] \\ &\quad - \left\{ \left[ct_1 - clv/(c^2 - v^2)\right] h + \left[vt_1 - lv^2/(c^2 - v^2)\right] i \right\} \\ &= clvh/(c^2 - v^2) + \left[l + lv^2/(c^2 - v^2)\right] i, \end{aligned}$$

and

$$\begin{aligned} |DE|^2 &= \left\{clvh/(c^2 - v^2) + \left[l + lv^2/(c^2 - v^2)\right] i\right\} \\ &\quad \times \left\{clvh/(c^2 - v^2) - \left[l + lv^2/(c^2 - v^2)\right] i\right\} \\ &= c^2 l^2 v^2 h^2/(c^2 - v^2)^2 - \left[l + lv^2/(c^2 - v^2)\right]^2 i^2 \end{aligned}$$

$$
\begin{aligned}
&= -c^2l^2v^2/(c^2-v^2)^2 + \left[l + lv^2/(c^2-v^2)\right]^2 \\
&= \frac{-c^2l^2v^2 + l^2(c^2-v^2)^2 + 2l^2v^2(c^2-v^2) + l^2v^4}{(c^2-v^2)^2} \\
&= \frac{l^2(-c^2v^2 + c^4 - 2c^2v^2 + v^4 + 2c^2v^2 - 2v^4 + v^4)}{(c^2-v^2)^2} \\
&= \frac{l^2(-c^2v^2 + c^4)}{(c^2-v^2)^2} \\
&= \frac{l^2c^2(c^2-v^2)}{(c^2-v^2)^2} \\
&= \frac{l^2c^2}{c^2-v^2} \\
&= \frac{l^2}{1-v^2/c^2}. \qquad (6.15)
\end{aligned}
$$

If we assume that the new complex number expressing DE as seen from B is $(DE)_B$, we have

$$(DE)_B = l_0 i. \qquad (6.16)$$

Thus, the equations become

$$
\begin{aligned}
|DE|^2 &= (DE)_B\overline{(DE)_B} \\
&= l_0 i(-l_0 i) \\
&= l_0^2. \qquad (6.17)
\end{aligned}
$$

From $l_0 > 0,\ l > 0$, (6.15), and (6.17), we can write

$$
\begin{aligned}
l_0^2 &= \frac{l^2}{1-v^2/c^2}, \\
l_0 &= \frac{l}{\sqrt{1-v^2/c^2}}, \\
l &= l_0\sqrt{1-v^2/c^2}. \qquad (6.18)
\end{aligned}
$$

Because (6.18) is the same as

$$l = l_0\sqrt{1-v^2/c^2}, \qquad (6.9)$$

which is obtained in special relativity, it is clear that the equation of length contraction in special relativity can be proven using the new complex plane.

In addition, l and l_0 are real numbers in (6.18), but as shown in (6.16), we live in a world where the distances are expressed with imaginary numbers. The correct distances are li and $l_0 i$, but l and l_0 can be observed.

7

The New Quaternion and World Distance

7.1 Definition of world distance in special relativity

In relativity theory, the interval of a world line is called world distance. In this chapter, the world distance is found using the new complex number. However, before coming to that, we explain how to find the world distance, which is written in standard books on relativity theory.

The light emitted at time $t = 0$ from origin O forms the spherical surface of radius ct at time t. The equation of the sphere is

$$x^2 + y^2 + z^2 = (ct)^2,$$

which can be rewritten as

$$(ct)^2 - x^2 - y^2 - z^2 = 0.$$

In special relativity, the value of

$$s^2 = (ct)^2 - x^2 - y^2 - z^2 \tag{7.1}$$

does not change, even if the Lorentz transformations

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{3.3}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \tag{3.4}$$

are substituted. Thus, (7.1) is defined as world distance.

Calculations can then be performed. The Lorentz transformations (3.3), (3.4) and

$$y' = y, \tag{3.5}$$

$$z' = z \tag{3.6}$$

are substituted for $(ct')^2 - x'^2 - y'^2 - z'^2$, and the equations become

$$\begin{aligned}
(ct')^2 - x'^2 - y'^2 - z'^2 &= c^2\left[\frac{t-(v/c^2)x}{\sqrt{1-v^2/c^2}}\right]^2 - \left[\frac{x-vt}{\sqrt{1-v^2/c^2}}\right]^2 - y^2 - z^2 \\
&= \frac{c^2t^2 - 2c^2t(v/c^2)x + c^2v^2x^2/c^4 - x^2 + 2vtx - v^2t^2}{1-v^2/c^2} - y^2 - z^2 \\
&= \frac{c^2t^2 - 2vtx + v^2x^2/c^2 - x^2 + 2vtx - v^2t^2}{1-v^2/c^2} - y^2 - z^2 \\
&= \frac{c^2t^2 + v^2x^2/c^2 - x^2 - v^2t^2}{1-v^2/c^2} - y^2 - z^2 \\
&= \frac{c^2t^2(1-v^2/c^2) - x^2(1-v^2/c^2)}{1-v^2/c^2} - y^2 - z^2 \\
&= c^2t^2 - x^2 - y^2 - z^2.
\end{aligned}$$

In other words, even if the Lorentz transformations are performed,

$$s^2 = (ct)^2 - x^2 - y^2 - z^2$$

is invariant.

In addition,

$$s^2 = -(ct)^2 + x^2 + y^2 + z^2 \tag{7.2}$$

is constant in the Lorentz transformations, even if the signs of $(ct)^2$ and $x^2+y^2+z^2$ are reversed. However, because the velocity $v(v_x,\ v_y,\ v_z)$ of a point mass does not exceed the velocity of light, s^2 becomes negative in (7.2). The reason is that

$$x^2 + y^2 + z^2 = v_x^2t^2 + v_y^2t^2 + v_z^2t^2 = (vt)^2 < (ct)^2.$$

Thus, generally,

$$s^2 = (ct)^2 - x^2 - y^2 - z^2 \tag{7.1}$$

is defined as the world distance s.

7.2 Derivation of world distance by the new quaternion

The theory that defines world distance in special relativity has some defects. Because the world distance s is defined by searching an invariant quantity under Lorentz transformations, it is unknown whether other definitions are possible. One might be able to choose other definitions of the world distance that fulfill the Lorentz invariance. In addition, it has not been mathematically proven which of (7.1) and (7.2) is correct. When the world distance is calculated using only the new quaternion, a unique formula of world distance is found.

As explained in Section 1.1, the magnitude squared of a complex number $A = a + bi$ on a flat surface is

$$\begin{aligned} |A|^2 &= A\overline{A} \\ &= (a+bi)(a-bi) \\ &= a^2 - abi + bai - b^2i^2 \\ &= a^2 + b^2. \end{aligned} \tag{1.1}$$

If the same calculation is performed on the new complex number $A = ah + bi$, we find

$$\begin{aligned} |A|^2 &= A\overline{A} \\ &= (ah+bi)(ah-bi) \\ &= a^2h^2 - b^2i^2 \\ &= -a^2 + b^2. \end{aligned} \tag{7.3}$$

The sign of a^2 in (7.3) is opposite to that of (1.1). In other words, (7.3) arises because space–time bends in reality, although the new complex plane written on paper looks like a flat surface.

In Hamilton's quaternion which extends the complex number in four dimensions, we have

$$A = a + bi + cj + dk,$$

$$\overline{A} = a - bi - cj - dk.$$

The algorithms are

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

Thus, the equations become

$$\begin{aligned} |A|^2 &= A\overline{A} \\ &= (a+bi+cj+dk)(a-bi-cj-dk) \\ &= a^2 - abi - acj - adk + bai - b^2i^2 - bcij - bdik \\ &\quad + caj - cbji - c^2j^2 - cdjk + dak - dbki - dckj - d^2k^2 \end{aligned}$$

$$
\begin{aligned}
&= a^2 - abi - acj - adk + abi + b^2 - bck + bdj \\
&\quad + acj + bck + c^2 - cdi + adk - bdj + cdi + d^2 \\
&= a^2 + b^2 + c^2 + d^2. \qquad (7.4)
\end{aligned}
$$

In the new quaternion, which extends the new complex number in four dimensions, we have

$$
\begin{aligned}
A &= ah + bi + cj + dk, \\
\overline{A} &= ah - bi - cj - dk.
\end{aligned}
$$

The algorithms

$$h^2 = -1,\ hi = ih,\ hj = jh,\ hk = kh$$

are added to the algorithms of Hamilton's quaternion, and the equations become

$$
\begin{aligned}
|A|^2 &= A\overline{A} \\
&= (ah + bi + cj + dk)(ah - bi - cj - dk) \\
&= a^2h^2 - abhi - achj - adhk + baih - b^2i^2 - bcij - bdik \\
&\quad + cajh - cbji - c^2j^2 - cdjk + dakh - dbki - dckj - d^2k^2 \\
&= -a^2 - abhi - achj - adhk + abhi + b^2 - bck + bdj \\
&\quad + achj + bck + c^2 - cdi + adhk - bdj + cdi + d^2 \\
&= -a^2 + b^2 + c^2 + d^2.
\end{aligned}
$$

In other words,

$$
\begin{aligned}
|A|^2 &= A\overline{A} \\
&= (ah + bi + cj + dk)(ah - bi - cj - dk) \\
&= -a^2 + b^2 + c^2 + d^2. \qquad (7.5)
\end{aligned}
$$

If (7.5) is compared with (7.4), we can see that the sign of a^2 is opposite as compared with that in (1.1) and (7.3). This result shows that the four-dimensional space–time expressed by the new quaternion is curved.

In four-dimensional space–time, the new quaternion, which expresses the location of the point mass at time t, which was at origin O at time $t = 0$, is $A = cth + xi + yj + zk$. Substituting this into (7.5), we have

$$
\begin{aligned}
|A|^2 &= A\overline{A} \\
&= (cth + xi + yj + zk)(cth - xi - yj - zk) \\
&= -(ct)^2 + x^2 + y^2 + z^2.
\end{aligned}
$$

Because $|A|$ is the world distance s, i.e., the distance between origin O and point A, the equation becomes

$$s^2 = -(ct)^2 + x^2 + y^2 + z^2. \tag{7.6}$$

Equation (7.6) is the mathematically calculated formula of the world distance. Because this distance in four-dimensional space–time was found mathematically, this formula is unique. In the last section, it was shown that

$$s^2 = (ct)^2 - x^2 - y^2 - z^2 \tag{7.1}$$

is the general definition of world distance in special relativity. However, (7.6) is the mathematically correct formula. Unlike (7.1), s^2 becomes negative in (7.6). There is no clear basis for choosing (7.1) or (7.6) in relativity theory texts. However, by the new quaternion, the mathematically correct definition of the world distance is

$$s^2 = -(ct)^2 + x^2 + y^2 + z^2. \tag{7.6}$$

7.3 Interpretation of the negativity of the square of the world distance

Using the new quaternion, we can explain what s becomes if s^2 is negative. In the calculation, by which (7.6) was obtained, the equation containing $h,\ i,\ j$, and k is

$$s^2 = (cth)^2 - (xi)^2 - (yj)^2 - (zk)^2. \tag{7.7}$$

If s is calculated from (7.7), under the condition that $(ct)^2 > x^2 + y^2 + z^2$, we find

$$\begin{aligned} s &= cth\sqrt{1 - \frac{(xi)^2 + (yj)^2 + (zk)^2}{(cth)^2}} \\ &= cth\sqrt{1 - \frac{x^2 + y^2 + z^2}{(ct)^2}}. \end{aligned} \tag{7.8}$$

The imaginary number h appearing in (7.8) signifies that the world distance s is the time part, and it is expressed by an imaginary number in four-dimensional space–time, just as it is in two-dimensional space-time.

Next, using the proper time, we again confirm that s is the time part. The coordinates $(cth,\ xi,\ yj,\ zk)$ of (7.7) are coordinates of point mass D as seen from observer A at rest. Because the coordinates of D as seen from observer B in uniform motion, are $(ct'h,\ x'i,\ y'j,\ z'k)$. Because s^2 is invariant in all frames, we can write

$$s^2 = (ct'h)^2 - (x'i)^2 - (y'j)^2 - (z'k)^2.$$

Therefore, we find

$$s = ct'h\sqrt{1 - \frac{x'^2 + y'^2 + z'^2}{(ct')^2}}. \tag{7.9}$$

If B coincides with D, time t' is the proper time τ of D and $x' = y' = z' = 0$ in (7.9). Thus, we find

$$s = c\tau h. \tag{7.10}$$

Equation (7.10) shows that the world distance s is an imaginary quantity obtained by multiplying proper time τ by c.

7.4 Least-squares theory

As explained in Section 7.1, the definition

$$s^2 = (ct)^2 - x^2 - y^2 - z^2$$

of world distance in special relativity becomes positive. Because the world distance s^2 becomes maximum when $x^2 + y^2 + z^2$ is a minimum, it is clear that s^2 is a maximum when the world line connecting two points is a straight line in four-dimensional space–time. Note that since the square of the distance between two points is a minimum when the world line is a straight line in two- and three-dimensional spaces, it is a unique property of the world distance that s^2 is a maximum when the world line is a straight line in four-dimensional space–time.

However, the formula of the world distance found mathematically by the new quaternion is

$$s^2 = -(ct)^2 + x^2 + y^2 + z^2.$$

In this formula, s^2 is a minimum when $x^2 + y^2 + z^2$ is a minimum. Thus, the square of the distance between two points is a minimum along a straight line in four-dimensional space–time, as it is in two- and three-dimensional spaces. This shows the accuracy of the formula

$$s^2 = -(ct)^2 + x^2 + y^2 + z^2$$

of the world distance by the new quaternion.

In addition, because s^2 is positive in special relativity, the world distance s is a real number. However, since s^2 is negative in the space–time theory using the new quaternion, the world distance s is an imaginary number. From this difference, although it is currently assumed that our universe lies in the four real axes in relativity theory, it is shown here using the new quaternion that our universe lies in the four imaginary axes in new quaternion space–time theory.

8

The New Quaternion and the World Distance of Light

8.1 Proof of the world distance of light by special relativity

According to special relativity, the world distance of light is 0. In other words, when light travels in four-dimensional space–time, the distance from the original point is 0. We regard this as very mysterious. However, if the new complex plane is used, it can be understood visually.

Before verification using the new complex plane, it is shown from the definition of the world distance in special relativity that the world distance of light is 0. This is the proof written in relativity theory texts.

The point mass, which leaves origin O at time $t = 0$ and moves in a straight line with uniform velocity $v(v_x,\ v_y,\ v_z)$, has a world distance s. Then, by the definition of world distance in special relativity, i.e.,

$$s^2 = (ct)^2 - x^2 - y^2 - z^2, \tag{7.1}$$

the equations become

$$\begin{aligned} s^2 &= (ct)^2 - x^2 - y^2 - z^2 \\ &= (ct)^2 - v_x^2t^2 - v_y^2t^2 - v_z^2t^2 \\ &= (ct)^2 - (vt)^2. \end{aligned} \tag{8.1}$$

Because the velocity v of light is c, from (8.1), the world distance s of light becomes as follows;

$$\begin{aligned} s^2 &= (ct)^2 - (ct)^2 \\ &= 0. \end{aligned}$$

In other words, no matter how far the light travels in four-dimensional space–time, the world distance is 0. Although it is a mysterious result, it can be easily understood using the new quaternion and the new complex plane in the next section.

8.2 Proof of the world distance of light by the new quaternion

Because the equation of the world line of light emitted in the x-direction from origin O at time $t = 0$ in four-dimensional space–time is $x = ct$, the coordinates are $(cth,\ cti,\ 0,\ 0)$ by substituting $x = ct$ in $(cth,\ xi,\ 0,\ 0)$. In addition, the new quaternion is $A = cth + cti$. Because $y = z = 0$, and thus considering only two-dimensional space–time, the world line of light becomes a set of points equidistant from the cth- and xi-axes in the new complex plane, as shown in Figure 8.1. We should be cautious not to say that this world line makes a 45° angle on these axes, because there is a possibility that the angle changes under rotation in the new complex plane, since this plane is curved, even though it appears to be flat. We will discuss this in Chapter 13.

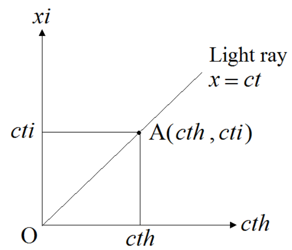


Figure 8.1

Because $|A|$ is the world distance s of light, we have

$$\begin{aligned} s^2 &= |A|^2 \\ &= A\overline{A} \\ &= (cth + cti)(cth - cti) \\ &= (cth)^2 - (cti)^2 \\ &= -(ct)^2 + (ct)^2 \\ &= 0. \end{aligned}$$

In other words, in the new complex plane, the distance between O and a point that is equidistant from the two coordinate axes is 0. This suggests that the new complex plane is a curved surface. In addition, it suggests that the new complex number and the new quaternion are mathematical quantities capable of describing curved space–time.

It is generally considered that a curved surface cannot be drawn on a flat paper, and the length of the world line cannot be directly measured on the paper. However, it can be calculated on a flat plane using the new complex number. This was proven in Section 5.1 and Section 6.3. The same method is also applied to the new quaternion. A node and a distance can be found through calculations using the equations of the world lines in four-dimensional space–time and the new quaternion.

The full theory of relativity includes both special and general relativity. The case in which the observer is at rest or undergoing linear uniform motion falls under special relativity, while the case in which the observer is accelerated falls under general relativity. It is assumed that space–time is flat in special relativity and it is curved in general relativity.

However, as shown here, space–time is curved even under uniform linear motion. Space–time bends doubly if the curvature by acceleration is also considered. This is a key difference between Einstein's relativity theory and the new quaternion space–time theory.

8.3 Interpretation of the zero world distance of light and the constancy of the velocity of light

As shown in Figure 8.1, in two-dimensional space–time, it can be said that the world line of light is equidistant from the two coordinate axes. However, a reverse aspect is possible. That is, the world line equidistant from the two coordinate axes is light. In the former aspect, light exists as an absolute object and its properties are explained. In the latter aspect, in many world lines, a world line equidistant from the two coordinate axes is light. In this aspect, the world line of a particle, which constitutes a substance, and the world line of light, have the same properties. The world line equidistant from the coordinate axes is light, and the world line, which is not equidistant, is the substance. If this theory is extended to four-dimensional space–time, light is not necessarily equidistant from each coordinate axis like it is in two-dimensional space–time. It passes only through the place whose world distance

$$s^2 = -(ct)^2 + x^2 + y^2 + z^2 \tag{7.6}$$

is zero. In other words, the difference between the substance and the light ray exists only at a particular place in four-dimensional space–time. The reason why light can be emitted in every direction in three-dimensional space is that the point $(ct,\ x,\ y,\ z)$, which satisfies $s = 0$ in (7.6), exists everywhere in the space.

In classical physics, light has properties of both waves and particles. On the other hand, in quantum mechanics, a substance exists as a wave and only becomes a particle via observation. Although it may seem odd that a substance can be a wave, it becomes clear when noted that the difference between the substance and a light ray is only a difference at a particular place in four-dimensional space–time. Because they are initially the same, it can be thought that light and substance have properties of both waves and particles. In this book, an axiom has been developed which states that the difference between substance and light is only a difference at a particular place in four-dimensional space–time, and discussions are carrried out below.

If this theory is continued further, we can consider the following. If the coefficient that changes time t into distance is c, then time is denoted as ct. Because the equation of the world line equidistant from the temporal and distance axes is $x = ct$ in this case, the velocity becomes c. This world line then corresponds to a light ray. In this theory, c is only a coefficient that changes time to distance. Because the path of the light ray lies on a set of points equidistant from the two coordinate axes, light has a velocity c. Then, the principle of the constancy of the velocity of light is no longer a principle or an axiom that cannot be proven. The principle is only a theorem drawn from the new quaternion.

If we assume that only four coordinate axes and world lines exist in four-dimensional space–time and that the world lines correspond to light rays or substances depending on the path, we can avoid the mistake that occurs when we assume that natural phenomena are only those that can be observed. According to the assumption that a natural phenomenon corresponds to a substance in special relativity, researchers have explained that mass becomes energy. However, they have not explained mass itself. In addition, in general relativity, the theory that mass bends space and produces gravity is based on the premise that mass exists independent of space. In other words, mass exists as an absolute and it is not assumed that it is a shadow of something else. In Chapter 18, we will show that the results of special relativity can be explained without contradiction if mass and energy are viewed as the time part of the unit world line and momentums as the space parts of the unit world line.

9

The Twin Paradox

9.1 Contents of the twin paradox

Because this book is not a commentary book of relativity theory but a book that proves physical laws using the new complex number, new quaternion, and new octonion, the paradoxes of special relativity are not a matter of concern in this book. However, the relations between the world line and force can be understood if the famous twin paradox is calculated using the new complex number.

At first, we explain contents of the twin paradox. We consider twins A and B. It is assumed that A is at rest and B departs using a rocket with uniform velocity v in the positive direction along the x-axis as seen from A. B accelerates suddenly after a sudden slowdown, turns around, and heads toward A with velocity $-v$. A and B synchronize their clocks at $t = 0$ when they depart. The times of the clocks of A and B are t and t', respectively, upon arrival. Because the formula of time dilation of the moving clock is

$$\tau = t\sqrt{1 - v^2/c^2}, \tag{6.3}$$

we have

$$t' = t\sqrt{1 - v^2/c^2}.$$

Since $1 - v^2/c^2 < 1$, the time of B is slower than that of A.

Switching to the reference frame of B, it is assumed that B is at rest and A leaves with velocity $-v$ in the negative direction along the x-axis as seen from B. It also seems that A accelerates suddenly after a sudden slowdown, turns around, and heads towards B with velocity v. If the times of the clocks of A and B are t and t', respectively, as mentioned above, then

$$t = t'\sqrt{1 - v^2/c^2}$$

because the clock of A moves. A ages slower than B.

This phenomenon is called the twin paradox. However, the premise that A and B are in the same conditions is incorrect. Since A is at rest, no force acts on A. However, a force acts on B if B accelerates suddenly after the sudden slowdown. On the other hand, no force acts on A when A appears to suddenly accelerate away from B after the sudden slowdown. B experiences a force during acceleration. Even if A seems to have accelerated suddenly after the sudden slowdown, A does not experinces a force. The forces acting on A and B appear to be indistinguishable if we consider only the change in the distance between A and B; however, the force acts only on B and not on A.

This difference is only explained conceptually in many books because the factor of the force cannot be included when the Lorentz transformations are used. However, the factor of the force can be added if we apply the new complex plane. The calculations are shown below.

9.2 Calculation in the frame of a stationary observer

First, we assume that B moves in the x-direction along A with uniform velocity v and then returns with velocity $-v$. The new complex plane emerges as shown in Figure 9.1. Acceleration a is constant.

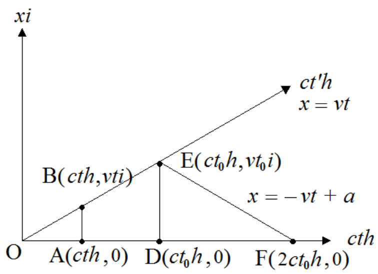


Figure 9.1

Although A is at rest, time passes. Thus, A moves in the cth-axis. In the reference frame of A, B moves in the straight line $x = vt$, i.e., the $ct'h$-axis. The world line of B becomes the straight line with gradient $-v$ at time t_0 and B meets A on the cth-axis at time $2t_0$. This node is called F. If the location of A at time t_0 is D, then the coordinates are $D(ct_0h,\ 0)$. In addition, if the node, where the velocity of B changes from v to $-v$, is E, then the coordinates are $E(ct_0h,\ vt_0i)$ by substituting $t = t_0$ for $x = vt$.

We assume that the elapsed times, during which the velocity of B changes from 0 to v and from v to $-v$, are relatively short so that these times can be neglected.

On the basis of the new complex number, point E is $ct_0h + vt_0i$. Using the definition of distances, the magnitude squared becomes as follows:

$$\begin{aligned} |OE|^2 &= (ct_0h + vt_0i)(ct_0h - vt_0i) \\ &= c^2t_0^2h^2 - v^2t_0^2i^2. \end{aligned} \tag{9.1}$$

In addition, if the new complex number showing the line-segment EF is written as (EF), then

$$\begin{aligned} (EF) &= F - E \\ &= 2ct_0h - (ct_0h + vt_0i) \\ &= ct_0h - vt_0i. \end{aligned}$$

Thus, we find

$$\begin{aligned} |EF|^2 &= (ct_0h - vt_0i)(ct_0h + vt_0i) \\ &= c^2t_0^2h^2 - v^2t_0^2i^2. \end{aligned} \tag{9.2}$$

From (9.1) and (9.2), we can write

$$|OE| + |EF| = 2\sqrt{c^2t_0^2h^2 - v^2t_0^2i^2}.$$

When $c > v$, the term inside the square root is

$$c^2t_0^2h^2 - v^2t_0^2i^2 = -c^2t_0^2 + v^2t_0^2 < 0.$$

Thus, using the axiom that place the negative sign outside of the square root when the quantity inside the square root is negative, we can write

$$|OE| + |EF| = 2ct_0h\sqrt{1 - v^2/c^2}. \tag{9.3}$$

In addition, if we use the new complex number $F = 2ct_0h$ of F, then we find

$$\begin{aligned} |OF|^2 &= (2ct_0h + 0i)(2ct_0h - 0i) \\ &= (2ct_0h)^2. \end{aligned}$$

Thus, the equation becomes

$$|OF| = 2ct_0h. \tag{9.4}$$

From (9.3) and (9.4), we find

$$|OE| + |EF| < |OF|.$$

This inequality implies that the elapsed time of B is shorter than that of A, and B is younger than A. The fact that $2ct_0h$ contains the imaginary number h in (9.4) indicates that $|OF|$ is the time part. Though $|OE| + |EF|$ appears to be longer than $|OF|$ in Figure 9.1, the line segment parallel to the cth- or x-axis is the longest and becomes shorter when it inclines from the coordinate axes, because the new complex plane curves in reality.

9.3 Calculation in the frame of a moving observer

Next, we consider the new complex plane, in which B is at rest and A moves. If we assume that a world line, which is the position of a point mass in four-dimensional space–time, travels like light with velocity v, then the connection of the world line between two points becomes a straight line. In other words, if no force acts, then the world line becomes a straight line. If a force acts, then the world line curves. If the force disappears, then the curve becomes a straight line. If the force continues to act, then the path is curved. We can say that the world line is curved when the force acts on the object. Also we can say that the curve of a world line is the force. The law of inertia states that a point mass moves in linear uniform motion if no force acts on it. However, if we consider the world line, then the law implies that no force results in a straight world line.

One may consider that because the space–time curves, a world line cannot be a straight line. However, in the new complex plane, a world line becomes a straight line if the equation of the world line is a linear function. It is thought that even if the world line in four-dimensional space–time is drawn using the new quaternion, the result is the same because the new quaternion itself contains the elements of curves. If the new quaternion is applied to a curved space–time, curves disappear and the world line becomes a straight line. This was proven in Section 5.1.

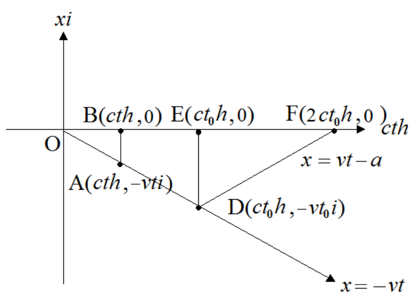


Figure 9.2

If the twin paradox is seen from the viewpoint of B, B is at rest and A separates from B with velocity $-v$ in the negative direction along the x-axis. Then, A accelerates suddenly after the sudden slowdown and returns to B with velocity v. We tend to think that the world line of B becomes the cth-axis because B is at rest, as shown in Figure 9.2. Also, we tend to think that the world line of A becomes a straight line $x = -vt$, changes to the straight line $x = vt - a$ at time t_0, and crosses the cth-axis at time $2t_0$. However, from the relation of the world line and force described previously, the force did not act in this figure, because B is on a straight line. The fact that the world line of A is curved means that force acted on A. This result conflicts with the precondition that the force acted only on B.

A graphical interpretation is not sufficient and can be misleading. Thus, we investigate using coordinate transformations. Because the multiplication of the new complex number implies a rotation about the origin O, a migration from point E to point D is denoted by a new complex number in Figure 9.1, which is drawn from the viewpoint of A.

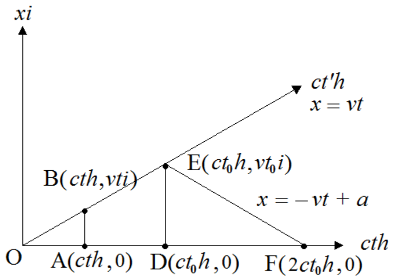


Figure 9.1

We assume that the denoted new complex number is H. Because the new complex numbers of D and E are

$$D = ct_0h,\ \ E = ct_0h + vt_0i$$

in Figure 9.1, using $EH = D$, we find

$$\begin{aligned}(ct_0h + vt_0i)H &= ct_0h, \\ H &= \frac{ct_0h}{ct_0h + vt_0i} \\ &= \frac{ct_0h(ct_0h - vt_0i)}{(ct_0h + vt_0i)(ct_0h - vt_0i)} \\ &= \frac{-c^2t_0^2 - cvt_0^2hi}{-c^2t_0^2 + v^2t_0^2}\end{aligned}$$

$$= \frac{c^2 + cvhi}{c^2 - v^2}$$
$$= \frac{1 + vhi/c}{1 - v^2/c^2}. \tag{9.5}$$

If the issue is considered using the coordinate transformation, then the fact that A is seen from B means that E moves to the location of D, which is in the rest frame expressed by the new complex number H, implying a rotation. The original D moves to another location. If we assume that E and D migrated by H are E' and D', respectively, and F migrates to F', then the new complex plane becomes as shown in Figure 9.3.

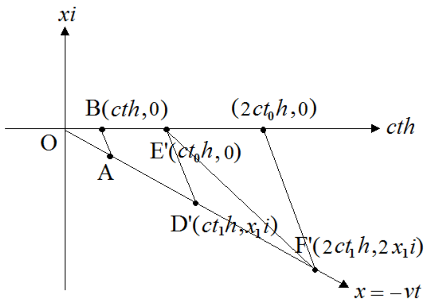


Figure 9.3

The coordinates of E' are the same as those of D in Figure 9.1: $E'(ct_0h,\ 0)$. Because the coordinates of F before migration are two times those of D, those of F' after the coordinate transformation by H are two times those of D'. Thus, if the coordinates of D' are $D'(ct_1h,\ x_1i)$, then those of F' are $F'(2ct_1h,\ 2x_1i)$. This is the mathematical proof that when no force acts on A, D' and F' are on the same world line. The world line of B is the cth axis at first and becomes a straight line from $E'(ct_0h,\ 0)$ at time t_0 and crosses the straight line OA at point F' at time $2t_0$. Using the new complex number H, which implies a rotation, the transformation, which moves D to D', is

$$DH = D'.$$

Thus, using (9.5), $D(ct_0h,\ 0)$, and $D'(ct_1h,\ x_1i)$, we have

$$ct_1h + x_1i = ct_0hH$$
$$= ct_0h(\frac{1 + vhi/c}{1 - v^2/c^2})$$
$$= \frac{ct_0h + vt_0h^2i}{1 - v^2/c^2}$$

$$= \frac{ct_0h - vt_0i}{1 - v^2/c^2}.$$

By comparing coefficients, we find

$$t_1 = \frac{t_0}{1 - v^2/c^2}, \tag{9.6}$$

$$x_1 = \frac{-vt_0}{1 - v^2/c^2}. \tag{9.7}$$

The slope of straight line OD' calculated using (9.6) and (9.7) is

$$\frac{x_1}{ct_1} = \frac{-vt_0}{ct_0} = -\frac{v}{c}.$$

Thus, the equation of the straight line OD' is

$$x = -\frac{v}{c}(ct) = -vt.$$

This result confirms that the initial relative velocity of A as seen from B is $-v$.

The length of the world line is calculated as follows. From (9.6) and (9.7), we have

$$\begin{aligned}
|OF'|^2 &= (2ct_1h + 2x_1i)(2ct_1h - 2x_1i) \\
&= 4c^2t_1^2h^2 - 4x_1^2i^2 \\
&= \frac{4c^2t_0^2h^2}{(1 - v^2/c^2)^2} - \frac{4v^2t_0^2i^2}{(1 - v^2/c^2)^2} \\
&= \frac{4c^2t_0^2h^2\left[1 - v^2i^2/(c^2h^2)\right]}{(1 - v^2/c^2)^2} \\
&= \frac{4c^2t_0^2h^2(1 - v^2/c^2)}{(1 - v^2/c^2)^2} \\
&= \frac{4c^2t_0^2h^2}{1 - v^2/c^2}.
\end{aligned}$$

Since $c > v$, we find

$$|OF'| = \frac{2ct_0h}{\sqrt{1 - v^2/c^2}}. \tag{9.8}$$

The imaginary number h shows that world distance $|OF'|$ is the temporal part.

If the new complex number describing line-segment $E'F'$ is written as $(E'F')$, then from (9.6) and (9.7), we find

$$\begin{aligned}
(E'F') &= F' - E' \\
&= 2ct_1h + 2x_1i - ct_0h \\
&= \frac{2ct_0h}{1 - v^2/c^2} - \frac{2vt_0i}{1 - v^2/c^2} - ct_0h
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2ct_0h}{1 - v^2/c^2} - \frac{2vt_0i}{1 - v^2/c^2} - \frac{ct_0h(1 - v^2/c^2)}{1 - v^2/c^2} \\
&= \frac{ct_0h(2 - 1 + v^2/c^2)}{1 - v^2/c^2} - \frac{2vt_0i}{1 - v^2/c^2} \\
&= \frac{ct_0h(1 + v^2/c^2)}{1 - v^2/c^2} - \frac{2vt_0i}{1 - v^2/c^2} \\
&= \frac{ct_0h(c^2 + v^2)}{c^2 - v^2} - \frac{2c^2vt_0i}{c^2 - v^2}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
|E'F'|^2 &= \left[\frac{ct_0h(c^2 + v^2)}{c^2 - v^2} - \frac{2c^2vt_0i}{c^2 - v^2}\right]\left[\frac{ct_0h(c^2 + v^2)}{c^2 - v^2} + \frac{2c^2vt_0i}{c^2 - v^2}\right] \\
&= \frac{c^2t_0^2h^2(c^2 + v^2)^2 - 4c^4v^2t_0^2i^2}{(c^2 - v^2)^2} \\
&= \frac{c^2t_0^2h^2}{(c^2 - v^2)^2}\left[(c^2 + v^2)^2 - 4c^2v^2\right] \\
&= \frac{c^2t_0^2h^2(c^2 - v^2)^2}{(c^2 - v^2)^2} \\
&= c^2t_0^2h^2.
\end{aligned}
$$

Finally, we find

$$|E'F'| = ct_0h. \tag{9.9}$$

In addition, because $E' = ct_0h$, we have

$$|OE'| = ct_0h. \tag{9.10}$$

From (9.9) and (9.10), we can write

$$|OE'| + |E'F'| = 2ct_0h.$$

Since

$$|OF'| = \frac{2ct_0h}{\sqrt{1 - v^2/c^2}}$$

from (9.8), we have

$$|OE'| + |E'F'| < |OF'|.$$

In other words, the elapsed time of B is shorter than that of A, and B is younger than A. This result is the same as that from the viewpoint of A. The twin paradox has now been solved.

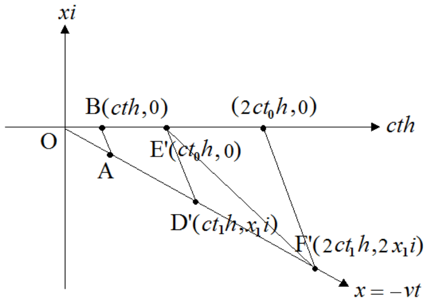


Figure 9.3

Though $|OE'|$ and $|E'F'|$ do not appear to be of the same length in Figure 9.3, they are the same when (9.9) and (9.10) are applied. This indicates that the new complex plane curves. In addition, because we performed the calculations using the new complex plane, we were able to obtain this result. Such a calculation cannot be performed in a regular Minkowski space-time diagram.

9.4 Curve of the world line and a force

In the discussion of the twin paradox as seen from B, some readers may consider calculating the new slope of B after point E', however, the world line of B is a straight line from the departure to arrival points, whereas the world line of A is curved. This is contrary to the axiom that if no force acts on A, then the world line is a straight line and if a force acts, then the world line curves. In other words, the curved world line of an accelerated point mass cannot be changed to a straight line by the coordinate transformations. This can be mathematically proven from the fact that a quadratic equation cannot be changed to a linear equation.

Consider Einstein's Gedankenexperiment. The people who are in an elevator, in which the supporting tie breaks, do not notice that they are falling, because they are falling at the same velocity as that of the elevator. In other words, it is the same as being at rest. However, because a free-falling object accelerates under the influence of gravity, the equation of the world line expresses a curve. It is clear that a force acts on the object and the objects are not at rest. The reference frames of the observer, who moves in linear uniform motion, are called inertial frames and the world line is a straight line. Thus, a free-fall movement of the elevator is not an inertial system if we consider using the world line. The fallacy of this Gedankenexperiment lies in considering the physical phenomenon using objects. We can avoid this error if the same phenomenon is considered using the world line. In this book, it is an axiom that if no force acts, then the world line becomes a straight line, which connects two points in four-dimensional space–time.

If we assume that observers A and B are on the same curved world line, then it is incorrect for them to think that they are at rest or move in linear uniform motion because of the constant distance between them. If we only consider the distance between two points, we tend to think that being at rest is relative, depending on the observer. However, if we think that the movement in a straight line between two points in four-dimensional space–time is that of an object at rest or moving in linear uniform motion, acceleration movement and linear uniform motion are distinguishable regardless of the observer. The most suitable example for understanding these concepts is the twin paradox.

10

The New Lorentz Transformations

10.1 Derivation of the new Lorentz transformations by the new quaternion

The Lorentz transformations using the coordinate transformation $\overline{A}/\,|A|$ by the new complex number are derived in the new complex plane in two-dimensional space–time. However, because we live in four-dimensional space–time, we assume that the coordinate transformation $\overline{A}/\,|A|$ by the new complex number is also realized by the new quaternion. Thus, we find the Lorentz transformations in four-dimensional space–time containing the y- and z-coordinates. Using the Lorentz transformations of special relativity, the y- and z-coordinates do not change and the equations are as follows:

$$y' = y, \tag{3.5}$$

$$z' = z. \tag{3.6}$$

However, we cannot find the same results using the coordinate transformation $\overline{A}/\,|A|$ by the new quaternion.

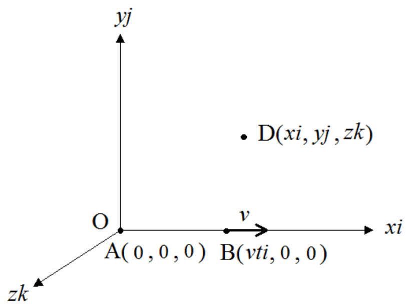


Figure 10.1

First, we considered the three-dimensional space, where time does not exist. As shown in Figure 10.1, we assume that observer A is at rest at origin O and observer B moves along a straight line with uniform velocity v in the positive direction along the x-axis. In addition, point D is at rest at point $(xi,\ yj,\ zk)$ in three-dimensional space. At time $t = 0$, A and B coincide at O. After t seconds, A is at $(0,\ 0,\ 0)$, B is at $(vti,\ 0,\ 0)$, and D is at $(xi,\ yj,\ zk)$ at time t. In four-dimensional space-time, the coordinates are $A(cth,\ 0,\ 0,\ 0)$, $B(cth,\ vti,\ 0,\ 0)$, and $D(cth,\ xi,\ yj,\ zk)$, and their new quaternions are $A = cth$, $B = cth + vti$, and $D = cth + xi + yj + zk$. If positions of A and B are drawn in a four-dimensional space–time diagram where the cth-axis is the horizontal axis and the spatial coordinate axes, which intersect perpendicularly with the cth-axis, are the xi-, yj-, and zk-axes, then the diagram becomes Figure 10.2. However, because the zk-axis, which is the fourth coordinate axis, cannot be drawn with a perspective on the flat surface, it is drawn with the dashed line. In addition, B is on a straight line $x = vt$ and since A, B, and D share the same time t, they are on the same flat plane perpendicular to the cth-axis.

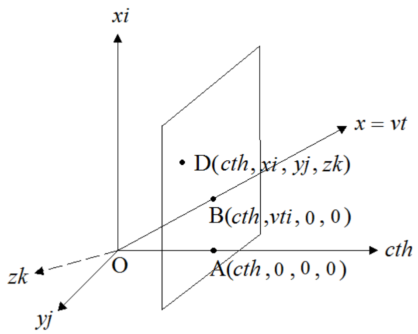


Figure 10.2

As the Lorentz transformations in the new complex plane are obtained, the equation of the coordinate transformation of D by B, i.e., how D is seen from B, is

$$\frac{D\overline{B}}{|B|}. \tag{10.1}$$

The new quaternions of B and D are substituted in (10.1) and calculations are performed. The algorithms of the new quaternion are as follows:

$$h^2 = i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,$$

$$hi = ih,\ hj = jh,\ hk = kh,$$

$$1/h = h/h^2 = -h.$$

If $c > v$, we have

$$\begin{aligned}
\frac{D\overline{B}}{|B|} &= \frac{(cth + xi + yj + zk)(cth - vti)}{\sqrt{(cth + vti)(cth - vti)}} \\
&= \frac{1}{\sqrt{c^2t^2h^2 - v^2t^2i^2}} \\
&\quad \times (c^2t^2h^2 - cvt^2hi + xcthi - xvti^2 + ycthj - yvtji + zcthk - zvtki) \\
&= \frac{1}{cth\sqrt{1 - v^2/c^2}} \\
&\quad \times (-c^2t^2 - cvt^2hi + xcthi + xvt + ycthj + yvtk + zcthk - zvtj) \\
&= \frac{1}{cth\sqrt{1 - v^2/c^2}} \\
&\quad \times \left[(-c^2t^2 + xvt) + (-cvt^2 + xct)hi + (ycth - zvt)j + (zcth + yvt)k\right] \\
&= \frac{1}{cth\sqrt{1 - v^2/c^2}} \\
&\quad \times [(-ct + xv/c)ct + (-vt + x)cthi + (yh - zv/c)ctj + (zh + yv/c)ctk] \\
&= \frac{1}{\sqrt{1 - v^2/c^2}} \\
&\quad \times [(-ct + xv/c)/h + (-vt + x)i + (yh - zv/c)j/h + (zh + yv/c)k/h] \\
&= \frac{1}{\sqrt{1 - v^2/c^2}} \\
&\quad \times [(ct - vx/c)h + (x - vt)i + (y + vzh/c)j + (z - vyh/c)k]\,. \qquad (10.2)
\end{aligned}$$

If the coordinates of D after the coordinate transformation by B are $(ct'h,\ x'i,\ y'j,\ z'k)$, the new quaternion is $ct'h + x'i + y'j + z'k$. If the coefficients are compared with (10.2), then the h part is

$$ct' = \frac{ct - vx/c}{\sqrt{1 - v^2/c^2}}.$$

Dividing both sides by c, we have

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}. \qquad (10.3)$$

The i, j, and k parts, respectively, are

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \qquad (10.4)$$

$$y' = \frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}}, \qquad (10.5)$$

$$z' = \frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}}. \tag{10.6}$$

Equations (10.3) and (10.4), respectively, coincide with

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{3.3}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \tag{3.4}$$

of the Lorentz transformations. However, (10.5) and (10.6), respectively, do not coincide with

$$y' = y, \tag{3.5}$$

$$z' = z \tag{3.6}$$

of the Lorentz transformations. From now on, (10.3), (10.4), (10.5), and (10.6) are referred to as new Lorentz transformations.

10.2 Differences between the new and original Lorentz transformations

Because $y' = y$ and $z' = z$ in the Lorentz transformations of special relativity, the y- and the z-coordinates do not change after the coordinate transformations because of the assumption of isotropy that space and time are uniform. Because our right-side space has the same property as the left-side space, even if we distinguish between the positive and negative x-direction, we can find the same equation

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}. \tag{3.4}$$

This is called the isotropy of space. However, the isotropy of space does not mean that a movement in the x-direction does not influence the y- and the z-coordinates. A movement in the x-direction changes the y- and z-coordinates, respectively, as can be understood by

$$y' = \frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}}, \tag{10.5}$$

$$z' = \frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}} \tag{10.6}$$

of the new Lorentz transformations. In other words, linear uniform motion in the x-direction bends the space in the y- and the z-directions. This is the new conclusion obtained from the coordinate transformations by the new quaternion.

There exist some texts in which the equation $y' = y$ of the Lorentz transformations is obtained by assuming that y' is a function of velocity v and coordinate y, i.e., $y' = f(v)y$. However, both y' and z' are functions of three variables, i.e., $y,\ z,$ and v, and the correct equations are $y' = f(y,\ z,\ v)$ and $z' = g(y,\ z,\ v)$ as understood by (10.5) and (10.6), respectively, if calculations are performed using the new quaternion. Because it is assumed that y' is a function of two variables, i.e., y and v, in special relativity, these results cannot be obtained. If this calculation is performed without assuming the property of space, then y' is a function of three variables, i.e., $y,\ z,$ and v. The same result can be obtained for z'.

Newtonian mechanics assumes absolute time and absolute space. This means that time and distance are constant regardless of an observer's velocity. Special relativity emerged by challenging these assumptions. Time and distance change with an observer's velocity in that theory. However, Einstein, who denied Newton's absolute time and absolute space, assumed the isotropy of space and concluded that

$$y' = y,$$

$$z' = z.$$

Because (10.3), (10.4), (10.5), and (10.6) are the formulae obtained without assuming the property of space, it is thought that they express the true property of space-time rather than the formulae of the Lorentz transformations, which assume the isotropy of space. The correctness of the new Lorentz transformations, which take the place of the Lorentz transformations, will be proven by considering that the space-time interval is invariant when using these new transformations in Section 10.3 and the velocity of light is constant using the new velocity transformation in Section 10.5. In addition, in Section 12.3, the implications of the imaginary number h in (10.5) and (10.6) are considered.

10.3 The new Lorentz transformations and world distance

To verify the correctness of the new Lorentz transformations, we examine whether the world distance becomes invariant under the new Lorentz transformations. As explained in Section 7.2, the equation of the world distance by the new quaternion is

$$s^2 = -(ct)^2 + x^2 + y^2 + z^2. \tag{7.6}$$

If we assume that the coordinates before the coordinate transformation are $(cth,\ xi,\ yj,\ zk)$ and those after the coordinate transformation are $(ct'h,\ x'i,\ y'j,\ z'k)$, then

we can write

$$-(ct)^2 + x^2 + y^2 + z^2 = -(ct')^2 + x'^2 + y'^2 + z'^2, \tag{10.7}$$

because the world distance is invariant using the coordinate transformation. We investigate whether the left side of (10.7) can be found by substituting the equations of the new Lorentz transformations for the right side of (10.7). The equations become

$$\begin{aligned} &-(ct')^2 + x'^2 + y'^2 + z'^2 \\ &= -c^2 \left[\frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}} \right]^2 + \left[\frac{x - vt}{\sqrt{1 - v^2/c^2}} \right]^2 \\ &\quad + \left[\frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}} \right]^2 + \left[\frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}} \right]^2 . \end{aligned} \tag{10.8}$$

Because the calculations are complex, we divide the right side of (10.8) into half. The first part is

$$\begin{aligned} &-c^2 \left[\frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}} \right]^2 + \left[\frac{x - vt}{\sqrt{1 - v^2/c^2}} \right]^2 \\ &= \frac{-c^2}{1 - v^2/c^2}(t^2 - 2vtx/c^2 + v^2x^2/c^4) + \frac{1}{1 - v^2/c^2}(x^2 - 2vtx + v^2t^2) \\ &= \frac{1}{1 - v^2/c^2}(-c^2t^2 + 2vtx - v^2x^2/c^2 + x^2 - 2vtx + v^2t^2) \\ &= \frac{1}{1 - v^2/c^2}(-c^2t^2 + v^2t^2 - v^2x^2/c^2 + x^2) \\ &= \frac{1}{1 - v^2/c^2}\left[-(1 - v^2/c^2)c^2t^2 + (1 - v^2/c^2)x^2 \right] \\ &= -(ct)^2 + x^2. \end{aligned} \tag{10.9}$$

Next, the second part of (10.8) is

$$\begin{aligned} &\left[\frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}} \right]^2 + \left[\frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}} \right]^2 \\ &= \frac{1}{1 - v^2/c^2}(y^2 + 2vyzh/c + v^2z^2h^2/c^2 + z^2 - 2vyzh/c + v^2y^2h^2/c^2) \\ &= \frac{1}{1 - v^2/c^2}(y^2 - v^2y^2/c^2 + z^2 - v^2z^2/c^2) \\ &= \frac{1}{1 - v^2/c^2}\left[(1 - v^2/c^2)y^2 + (1 - v^2/c^2)z^2 \right] \\ &= y^2 + z^2. \end{aligned} \tag{10.10}$$

From (10.8), (10.9), and (10.10), we find

$$-(ct')^2 + x'^2 + y'^2 + z'^2 = -(ct)^2 + x^2 + y^2 + z^2.$$

Thus, the coordinate transformations under the new Lorentz transformations make the world distance invariant. In other words, the y'- and z'-coordinates of the new Lorentz transformations, i.e.,

$$y' = \frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}}, \tag{10.5}$$

$$z' = \frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}} \tag{10.6}$$

do not produce a mathematical contradiction.

10.4 Transformation of velocities under Lorentz transformations and the constancy of the velocity of light

Next, we investigate whether the constancy of the velocity of light is maintained under the new Lorentz transformations. However, before that, the well-known Lorentz transformations for velocity are explained using special relativity.

First, we explain the differential form of velocity for those who have not learned the calculus. If the velocity does not change, the relation between the distance x, time t, and velocity v is $v = x/t$. However, if the velocity changes, the general formula for velocity becomes

$$v = \frac{dx}{dt}. \tag{10.11}$$

Here, dx and dt imply an infinitesimally small distance and small period of time, respectively. It is thought that even if the velocity changes with time, it is constant between each infinitesimally small period of time. By dividing dx by dt, (10.11) is considered to be the general formula of the velocity. It is enough to think that d means infinitesimally small.

Because the equations of the Lorentz transformations are as follows:

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{3.3}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \tag{3.4}$$

$$y' = y, \tag{3.5}$$

$$z' = z, \tag{3.6}$$

the differentials are

$$dt' = \frac{dt - (v/c^2)dx}{\sqrt{1 - v^2/c^2}}, \tag{10.12}$$

$$dx' = \frac{dx - vdt}{\sqrt{1 - v^2/c^2}}, \tag{10.13}$$

$$dy' = dy, \tag{10.14}$$

$$dz' = dz. \tag{10.15}$$

It is assumed that the velocity of point mass D is V as seen from observer A at rest and is V' as seen from observer B who is moving with constant velocity v. We assume that the coordinates belonging to V and V' are $(V_x,\ V_y,\ V_z)$ and $(V'_x,\ V'_y,\ V'_z)$, respectively. Thus, from (10.12) and (10.13), we find

$$\begin{aligned} V'_x &= \frac{dx'}{dt'} \\ &= \frac{dx - vdt}{dt - (v/c^2)dx}. \end{aligned}$$

By dividing the numerator and denominator on the right side by dt, we have

$$\begin{aligned} V'_x &= \frac{dx/dt - v}{1 - (v/c^2)dx/dt} \\ &= \frac{V_x - v}{1 - (v/c^2)V_x}. \end{aligned} \tag{10.16}$$

Similarly, from (10.12), (10.14), and (10.15), we find

$$\begin{aligned} V'_y &= \frac{dy'}{dt'} \\ &= \frac{dy\sqrt{1 - v^2/c^2}}{dt - (v/c^2)dx} \\ &= \frac{dy/dt\sqrt{1 - v^2/c^2}}{1 - (v/c^2)dx/dt} \\ &= \frac{V_y\sqrt{1 - v^2/c^2}}{1 - (v/c^2)V_x}, \end{aligned} \tag{10.17}$$

$$\begin{aligned} V'_z &= \frac{dz'}{dt'} \\ &= \frac{dz\sqrt{1 - v^2/c^2}}{dt - (v/c^2)dx} \\ &= \frac{dz/dt\sqrt{1 - v^2/c^2}}{1 - (v/c^2)dx/dt} \end{aligned}$$

$$= \frac{V_z\sqrt{1-v^2/c^2}}{1-(v/c^2)V_x}. \tag{10.18}$$

Equations (10.16), (10.17), and (10.18) are the transformation equations of the velocities obtained from the Lorentz transformations.

In the case that observed subject D is light emitted in the x-direction from A at rest, we calculate how the velocity of the light is observed as seen from B, which is moving in the x-direction with a constant velocity v. By substituting the velocity components of the light, i.e., $V_x = c$, $V_y = 0$, and $V_z = 0$, in (10.16), (10.17), and (10.18), we have

$$\begin{aligned} V'_x &= \frac{c-v}{1-(v/c^2)c} \\ &= \frac{c-v}{1-v/c} \\ &= \frac{c(c-v)}{c-v} \\ &= c, \end{aligned} \tag{10.19}$$

$$\begin{aligned} V'_y &= \frac{0}{1-(v/c^2)c} \\ &= 0, \end{aligned} \tag{10.20}$$

$$\begin{aligned} V'_z &= \frac{0}{1-(v/c^2)c} \\ &= 0. \end{aligned} \tag{10.21}$$

Equation (10.19) shows that the velocity of light is always a constant value c, regardless of the velocity v of observer B. This is a confirmation of the constancy of the velocity of light under the Lorentz transformations.

It is not correct here to use the word proof instead of confirmation because it is mathematically natural to obtain the constancy of the velocity of light from the transformation equations of the velocity using the Lorentz transformations. After all, the Lorentz transformations are obtained on the basis of the assumption that the velocity of light is constant as explained in Section 3.1. Therefore, (10.19) is the confirmation of the correctness of the transformation equations of the velocity calculated under the Lorentz transformations and it is not the proof of the constancy of the velocity of light. Thus, it should not be thought that the constancy of the velocity of light has been proven using this calculation. Because verification is impossible but a contradiction does not occur, the constancy of the velocity of light is considered to be correct. In mathematical terms, it is an axiom. However, in this book, it is established that the constancy of the velocity of light is not an axiom; it is rather a theorem obtained from other axioms in Section 8.3.

10.5 Transformations of velocities under the new Lorentz transformations and the constancy of the velocity of light

We will confirm that the new Lorentz transformations are the correct equations of transformation, if we also obtain the constancy of the velocity of light from the transformation equations of the velocities calculated using the new Lorentz transformations and the new quaternion.

As explained in Section 10.1, the equations of the new Lorentz transformations, which are obtained using the coordinate transformation $\overline{A}/\ |A|$ by the new quaternion, are

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{10.3}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \tag{10.4}$$

$$y' = \frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}}, \tag{10.5}$$

$$z' = \frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}}. \tag{10.6}$$

The differentials are

$$dt' = \frac{dt - (v/c^2)dx}{\sqrt{1 - v^2/c^2}}, \tag{10.21}$$

$$dx' = \frac{dx - vdt}{\sqrt{1 - v^2/c^2}}, \tag{10.22}$$

$$dy' = \frac{dy + (v/c)dzh}{\sqrt{1 - v^2/c^2}}, \tag{10.23}$$

$$dz' = \frac{dz - (v/c)dyh}{\sqrt{1 - v^2/c^2}}. \tag{10.24}$$

If we consider the assumptions similar to those in the Lorentz transformations, from (10.21), (10.22), (10.23), and (10.24), we find

$$\begin{aligned} V'_x &= \frac{dx'}{dt'} \\ &= \frac{dx - vdt}{dt - (v/c^2)dx} \end{aligned}$$

$$
\begin{aligned}
&= \frac{dx/dt - v}{1-(v/c^2)dx/dt} \\
&= \frac{V_x - v}{1-(v/c^2)V_x}, \qquad (10.25) \\
V'_y &= \frac{dy'}{dt'} \\
&= \frac{dy + (v/c)dzh}{dt - (v/c^2)dx} \\
&= \frac{dy/dt + (v/c)\ hdz/dt}{1-(v/c^2)dx/dt} \\
&= \frac{V_y + (v/c)V_z h}{1-(v/c^2)V_x}, \qquad (10.26) \\
V'_z &= \frac{dz'}{dt'} \\
&= \frac{dz - (v/c)dyh}{dt - (v/c^2)dx} \\
&= \frac{dz/dt - (v/c)hdy/dt}{1-(v/c^2)dx/dt} \\
&= \frac{V_z - (v/c)V_y h}{1-(v/c^2)V_x}. \qquad (10.27)
\end{aligned}
$$

These are the transformation equations for velocity using the new Lorentz transformations.

Next, we investigate whether the constancy of the velocity of light is obtained using (10.25), (10.26), and (10.27). If observed subject D is light emitted in the x-direction from A at rest, the velocities of the light measured by A are $V_x = c,\ V_y = 0$, and $V_z = 0$. Thus, if $V_x = c,\ V_y = 0$, and $V_z = 0$ are substituted in (10.25), (10.26), and (10.27), we have

$$
\begin{aligned}
V'_x &= \frac{c - v}{1-(v/c^2)c} \\
&= \frac{c-v}{1-v/c} \\
&= \frac{c(c-v)}{c-v} = c, \qquad (10.28) \\
V'_y &= \frac{0}{1-(v/c^2)c} = 0, \\
V'_z &= \frac{0}{1-(v/c^2)c} = 0.
\end{aligned}
$$

Equation (10.28) indicates that the velocity of light is always a constant value c, regardless of velocity v of observer B, which is confirmed by the above results using the new Lorentz transformations.

10.6 Independency of the imaginary number *h*

Because the velocity transformations using the new Lorentz transformations have been explained, we prove the independency of the imaginary number h in this section. In Section 3.4, to make the coordinate transformations by the complex number similar to the Lorentz transformations, calculations were performed using cth by multiplying ct with the imaginary number h. By performing these calculations, the signs of c^2 of

$$t' = \frac{t + (v/c^2)x}{\sqrt{1 + v^2/c^2}}, \tag{3.9}$$

$$x' = \frac{x - vt}{\sqrt{1 + v^2/c^2}} \tag{3.10}$$

changed from (+) to (-) and the Lorentz transformations could be found.

Because the fourth imaginary number h was introduced to change velocity c of light into ch as mentioned above, a question arises whether ch itself has any phisical meaning. In other words, can h be used independently? This independency of the imaginary number h can be proven using the velocity transformation equations (10.25), (10.26), and (10.27) using the new Lorentz transformations.

In addition, this was not proven in Section 3.4 as the velocity transformations using the new Lorentz transformations are explained first in this chapter.

We calculate how the velocity of the light, which is emitted along the y-direction from observer A at rest, is observed from observer B, who moves in the x-direction with velocity v. If $V_x = 0$, $V_y = c$, and $V_z = 0$ are substituted in the velocity transformation equations using the new Lorentz transformations, i.e.,

$$V'_x = \frac{V_x - v}{1 - (v/c^2)V_x}, \tag{10.25}$$

$$V'_y = \frac{V_y + (v/c)V_z h}{1 - (v/c^2)V_x}, \tag{10.26}$$

$$V'_z = \frac{V_z - (v/c)V_y h}{1 - (v/c^2)V_x}, \tag{10.27}$$

we find

$$\begin{aligned} V'_x &= \frac{0 - v}{1 - (v/c^2)0} \\ &= -v, \end{aligned} \tag{10.29}$$

$$\begin{aligned} V'_y &= \frac{c + (v/c)0h}{1 - (v/c^2)0} \\ &= c, \end{aligned} \tag{10.30}$$

$$V_z' = \frac{0 - (v/c)ch}{1 - (v/c^2)0}$$
$$= -vh. \tag{10.31}$$

In (10.31), h has been separated from c and combined with v. In other words, the imaginary number h is a number independent of c. Therefore, there is no problem if we multiply or divide by h alone.

The correctness of (10.29), (10.30), and (10.31) is proven because they satisfy the principle of the velocity of light from

$$\sqrt{V_x'^2 + V_y'^2 + V_z'^2} = \sqrt{(-v)^2 + c^2 + (-vh)^2}$$
$$= \sqrt{v^2 + c^2 - v^2}$$
$$= c.$$

In addition, (10.29), (10.30), and (10.31) differ from the conclusions obtained from the Lorentz transformations. If $V_x = 0$, $V_y = c$, and $V_z = 0$ are substituted for the velocity transformations

$$V_x' = \frac{V_x - v}{1 - (v/c^2)V_x}, \tag{10.16}$$
$$V_y' = \frac{V_y\sqrt{1 - v^2/c^2}}{1 - (v/c^2)V_x}, \tag{10.17}$$
$$V_z' = \frac{V_z\sqrt{1 - v^2/c^2}}{1 - (v/c^2)V_x} \tag{10.18}$$

obtained from the Lorentz transformations, then we find

$$V_x' = -v,$$
$$V_y' = c\sqrt{1 - v^2/c^2},$$
$$V_z' = 0.$$

These results differ from (10.29), (10.30), and (10.31). However, also in this case, they agree with the principle of the velocity of light because

$$\sqrt{V_x'^2 + V_y'^2 + V_z'^2} = \sqrt{(-v)^2 + c^2 - v^2}$$
$$= c.$$

As proven in Section 10.5, the same conclusions can be obtained if light is emitted in the direction of moving observer B from observer A at rest under the Lorentz and new Lorentz transformations. If light is emitted at a right angle in the direction along the movement of B, as seen from observer A at rest, the principle of the velocity of light is realized in the Lorentz and new Lorentz transformations as mentioned above. However, although V_x' is same, the values of V_y' and V_z' are different. In the Lorentz transformations, the vertical component of the velocity of light, i.e.,

$$V_y' = c\sqrt{1 - v^2/c^2},$$

decreases as v approaches c. However, V_y' is the constant value c in the new Lorentz transformations. This difference is investigated in Chapter 16.

10.7 The new Lorentz transformations and inverse transformations

Using the third proof, we show that the new Lorentz transformations are mathematically consistent, which has been already proven by two methods, i.e., invariance of the world distance using coordinate transformations and the constancy of the velocity of light. The reader may feel that the third proof is unnecessary; however, because the new quaternion is the new mathematics, which does not exist anywhere, it is necessary to verify its correctness by various methods.

First, we consider the Lorentz transformations. The equations for t' and x' of the Lorentz transformations are as follows:

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{3.3}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}. \tag{3.4}$$

If observer B moves with velocity v as seen from observer A, then A moves with velocity $-v$ as seen from B. Thus, the equation can be rewritten by substituting $v \to -v$, $t' \to t$, $t \to t'$, and $x \to x'$ in (3.3). This becomes the Lorentz transformation under the condition that B is at rest and A moves with velocity $-v$. This is called inverse transformation. If we substitute $v \to -v$, $t' \to t$, $t \to t'$, and $x \to x'$ in (3.3), we find

$$t = \frac{t' + (v/c^2)x'}{\sqrt{1 - v^2/c^2}}. \tag{10.32}$$

If the equation of this inverse transformation can be obtained by subtracting x from (3.3) and (3.4), we can see that the correctness of the Lorentz transformation is proven.

By multiplying both sides of (3.4) by (v/c^2), we have

$$(v/c^2)x' = \frac{(v/c^2)x - (v^2/c^2)t}{\sqrt{1 - v^2/c^2}}. \tag{10.33}$$

By adding both sides of (3.3) and (10.33), we have

$$\begin{aligned}(v/c^2)x' + t' &= \frac{t - (v^2/c^2)t}{\sqrt{1 - v^2/c^2}} \\ &= t\sqrt{1 - v^2/c^2}.\end{aligned}$$

Then we find

$$t = \frac{t' + (v/c^2)x'}{\sqrt{1 - v^2/c^2}}. \tag{10.34}$$

Equation (10.34) is the same as (10.32). In other words, (10.34) is the inverse transformation equation. The fact that this expression is obtained by subtracting x from (3.3) and (3.4) shows that (3.3) and (3.4) are mathematically consistent.

Similarly, if we can show that the equation obtained by subtracting z from the equations for y' and z' of the new Lorentz transformations, i.e.,

$$y' = \frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}}, \tag{10.5}$$

$$z' = \frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}}, \tag{10.6}$$

agrees with the equation obtained by substituting $v \to -v$, $y' \to y$, $y \to y'$, and $z \to z'$ in (10.5), then (10.5) and (10.6) are confirmed to be mathematically consistent.

By multiplying both sides of (10.6) by $-(v/c)h$, we have

$$-(v/c)z'h = \frac{-(v/c)zh - (v^2/c^2)y}{\sqrt{1 - v^2/c^2}}. \tag{10.35}$$

By adding both sides of (10.5) and (10.35) and subtracting $(v/c)zh$, we find that

$$\begin{aligned}y' - (v/c)z'h &= \frac{y - (v^2/c^2)y}{\sqrt{1 - v^2/c^2}} \\ &= y\sqrt{1 - v^2/c^2}.\end{aligned}$$

Thus, we have

$$y = \frac{y' - (v/c)z'h}{\sqrt{1 - v^2/c^2}}. \tag{10.36}$$

Equation (10.36) is the equation in which we substituted $v \to -v$, $y' \to y$, $y \to y'$, and $z \to z'$ in (10.5). In other words, it is the equation of the inverse transformation of (10.5). This result shows that (10.5) and (10.6) are not mathematically contradictory.

As mentioned above, it has been proven that no contradiction originates from the new Lorentz transformations using invariance of the coordinate transformation of the world distance, the constancy of the velocity of light, and the inverse transformation. The structure of space–time, which has not been known until now, is suggested by the fact that the transformation equations of y' and z' of the new Lorentz transformations contain the imaginary number h unlike the Lorentz transformations. We will consider it in Chapter 11.

10.8 General formulae of the new Lorentz transformations

When we obtained the new Lorentz transformations in Section 10.1, the direction of velocity v of observer B was limited to the x-direction along observer A. Because the coordinate transformation $\overline{A}/|A|$ by the new quaternion is a mathematical calculation, we can obtain the general formulae of the new Lorentz transformations in the case that the direction of velocity v of B is arbitrary. On the other hand, because researchers consider light graphically in special relativity, they cannot imagine the general formulae of the Lorentz transformations. In fact, there is no book of the theory of relativity in which the general formulae are written.

To obtain the general formulae describing the new Lorentz transformations, it is enough for us to perform a coordinate transformation by changing the coordinates $(cth,\ vti,\ 0,\ 0)$ of B to $(cth,\ v_xti,\ v_ytj,\ v_ztk)$ in Section 10.1. First, we consider a three-dimensional space. we assume that observer A is at rest at the origin O and observer B moves in a straight line with constant velocity v. The components of velocity v are $(v_xi,\ v_yj,\ v_zk)$. In addition, the observed point D is at rest at $(xi,\ yj,\ zk)$, and A and B are coincident at the origin O at time $t = 0$. After t seconds, A is at $(0,\ 0,\ 0)$ at time t, B is at $(v_xti,\ v_ytj,\ v_ztk)$ at time t, and D is at $(xi,\ yj,\ zk)$ at time t so that in four-dimensional space-time, the coordinates bescome $A(cth,\ 0,\ 0,\ 0)$, $B(cth,\ v_xti,\ v_ytj,\ v_ztk)$, and $D(cth,\ xi,\ yj,\ zk)$. Their new quaternions are

$$A = cth,\ B = cth + v_xti + v_ytj + v_ztk,\ D = cth + xi + yj + zk.$$

The equation of the coordinate transformation of D by B, i.e., the equation showing how D is seen by B, is $D\overline{B}/|B|$. If $c > v$, we find that

$$\frac{D\overline{B}}{|B|} = \frac{(cth + xi + yj + zk)(cth - v_xti - v_ytj - v_ztk)}{\sqrt{(cth + v_xti + v_ytj + v_ztk)(cth - v_xti - v_ytj - v_ztk)}}$$

$$= \frac{1}{\sqrt{c^2t^2h^2 - v_x^2t^2i^2 - v_y^2t^2j^2 - v_z^2t^2k^2}}$$
$$\times [cth(cth - v_xti - v_ytj - v_ztk) + xi(cth - v_xti - v_ytj - v_ztk)$$
$$+ yj(cth - v_xti - v_ytj - v_ztk) + zk(cth - v_xti - v_ytj - v_ztk)]$$
$$= \frac{1}{\sqrt{c^2t^2h^2 - v_x^2t^2i^2 - v_y^2t^2j^2 - v_z^2t^2k^2}}$$
$$\times [-c^2t^2 - cv_xt^2hi - cv_yt^2hj - cv_zt^2hk + ctxhi + v_xtx - v_ytxk + v_ztxj$$
$$+ ctyhj + v_xtyk + v_yty - v_ztyi + ctzhk - v_xtzj + v_ytzi + v_ztz]$$
$$= \frac{1}{cth\sqrt{1 - v^2/c^2}}$$
$$\times [(-c^2t + v_xx + v_yy + v_zz)t + (x - v_xt)cthi + (y - v_yt)cthj + (z - v_zt)cthk$$
$$+ (v_yz - v_zy)ti + (v_zx - v_xz)tj + (v_xy - v_yx)tk]$$
$$= \frac{1}{\sqrt{1 - v^2/c^2}}$$
$$\times [(ct - v_xx/c - v_yy/c - v_zz/c)h + (x - v_xt)i + (y - v_yt)j + (z - v_zt)k$$
$$+ (v_zy/c - v_yz/c)hi + (v_xz/c - v_zx/c)hj + (v_yx/c - v_xy/c)hk]. \quad (10.37)$$

If we assume that the coordinates of D after the coordinate transformation by B are $(ct'h,\ x'i,\ y'j,\ z'k)$, then the new quaternion becomes $ct'h + x'i + y'j + z'k$. Thus, by comparing the coefficients, the h part is

$$ct' = \frac{1}{\sqrt{1 - v^2/c^2}}(ct - v_xx/c - v_yy/c - v_zz/c).$$

By dividing both sides by c, we have

$$t' = \frac{1}{\sqrt{1 - v^2/c^2}}(t - v_xx/c^2 - v_yy/c^2 - v_zz/c^2). \quad (10.38)$$

The $i,\ j$, and k parts are

$$x' = \frac{1}{\sqrt{1 - v^2/c^2}}(x - v_xt + v_zyh/c - v_yzh/c), \quad (10.39)$$

$$y' = \frac{1}{\sqrt{1 - v^2/c^2}}(y - v_yt + v_xzh/c - v_zxh/c), \quad (10.40)$$

$$z' = \frac{1}{\sqrt{1 - v^2/c^2}}(z - v_zt + v_yxh/c - v_xyh/c), \quad (10.41)$$

respectively. (10.38), (10.39), (10.40), and (10.41) are the general formulae of the new Lorentz transformations.

The case that B moves in the x-direction corresponds the case of $v_y = v_z = 0$ in the above equations. Thus, if $v_y = v_z = 0$ are substituted in (10.38), (10.39), (10.40), and (10.41), we find

$$t' = \frac{t - v_x x/c^2}{\sqrt{1 - v^2/c^2}}, \tag{10.42}$$

$$x' = \frac{x - v_x t}{\sqrt{1 - v^2/c^2}}, \tag{10.43}$$

$$y' = \frac{y + v_x zh/c}{\sqrt{1 - v^2/c^2}}, \tag{10.44}$$

$$z' = \frac{z - v_x yh/c}{\sqrt{1 - v^2/c^2}}. \tag{10.45}$$

Equations (10.42), (10.43), (10.44), and (10.45) agree with the new Lorentz transformations, i.e., (10.3), (10.4), (10.5), and (10.6), respectively, which were obtained in Section 10.1.

It may seem odd that the signs of the h terms are opposite in the equations for y' and z' of the new Lorentz transformations, i.e.,

$$y' = \frac{y + v_x zh/c}{\sqrt{1 - v^2/c^2}}, \tag{10.5}$$

$$z' = \frac{z - v_x yh/c}{\sqrt{1 - v^2/c^2}}, \tag{10.6}$$

which were obtained in Section 10.1. However, since there is no polarization in the general formulae (10.40) and (10.41) of the new Lorentz transformations, there is no mathematical contradiction. If the conditions $v_y = v_z = 0$ are imposed in (10.40) and (10.41), polarizations occur in y' and z', respectively. We tend to think that the selection of the y- and z-axes must be arbitrary and neither axis is dominant. Therefore, y' and z' must not be polarized. However, after observer A determines the directions of the y- and z-axes, the y'- and z'-axes become unique and polarizations occur.

11

Double Structure of Four-Dimensional Space-Time

11.1 The algebraic theorem

A wise reader has probably noticed that, because the new quaternion consists of five numbers, one real and four imaginary numbers designated h, i, j, and k, it must be a five-element number. An algebraic theorem states that the algebraic calculations of addition, subtraction, multiplication, and division can be operated only on real numbers, complex numbers, quaternions, and octonions. In other words, five-element numbers are mathematically implausible.

We explain the above algebraic theorem with an example. If a complex number comprises two real numbers a and b and an imaginary number i, no new number forms can exist; for example, multiplying i by b yields new imaginary number bi. Even if we multiply two or more imaginary numbers, the result is either a real number or an imaginary number; for instance, $bi \times i = -b$ and $bi \times i \times i = -bi$. In other words, regardless of the calculation, real and imaginary are the only possible number forms, and complex numbers are regarded as two-element numbers. The same conclusion is drawn from Hamilton's quaternions. The quaternion comprises one real number and three imaginary numbers, denoted i, j, and k, where

$$i^2 = j^2 = k^2 = -1,$$

$$ij = k,\ jk = i,\ ki = j,$$

$$ijk = k^2 = -1.$$

All calculations performed on quaternions yield one real number and three imaginary numbers i, j, and k. Therefore, quaternions are four-element numbers.

The new quaternion contains three numbers hi, hj, and hk in addition to five numbers (one real and four imaginary, designated h, i, j, and k). If hi, hj, and

hk remain after a set of calculations, they cannot be reduced to easier forms. Thus, they are independent elements in the new quaternion. In other words, because the new quaternion consists of eight fundamental numbers, one real number and seven numbers h, i, j, k, hi, hj, and hk, it is called an octonion. Furthermore, hi, hj, and hk are obtained by multiplying two imaginary numbers whose square is, e.g., $(hi)(hi) = h^2i^2 = 1$. Thus, they are considered as new real numbers. In other words, the new quaternion is an octonion composed of four real and four imaginary numbers.

The existing octonions, known as Cayley number, operate in flat space–time. In contrast, the new quaternion operates in curved space–time. In addition, unlike the new quaternion, Cayley numbers consist of one real number and seven imaginary numbers. Considering this difference, we hereafter refer to the new numbers as new octonions. Because the new octonion is the highest number class admitted by the algebraic theorem, it may be able to calculate all phenomena in curved four-dimensional space–time.

11.2 The octonion

Before describing the properties of the new octonion, we introduce Graves' octonion discovered by John T. Graves in 1844. For details of Graves' octonion, the reader is referred to John H. Conway and Derec A. Smith's work *On Quaternions and Octonions*. Graves was a friend of Hamilton, who just a year earlier had discovered the quaternion. Thus, the quaternion and octonion were discovered essentially at the same time. Graves informed Hamilton of his discovery and ongoing investigation of the octonion by letter. However, Arthur Cayley formally published his study in 1845. Although Hamilton communicated to the academic journal that Graves was its first discoverer, octonions had become widely known as Cayley numbers, a name that has persisted today.

The octonion consists of one real number and seven imaginary numbers. Assuming that a and b_n are real numbers and i_n is an imaginary number, the octonion can be written as

$$a + b_1i_1 + b_2i_2 + b_3i_3 + b_4i_4 + b_5i_5 + b_6i_6 + b_7i_7.$$

Its algorithms are as follows:

$$i_n^2 = -1,$$

$$i_ni_{n+1} = i_{n+3} = -i_{n+1}i_n,$$

$$i_{n+1}i_{n+3} = i_n = -i_{n+3}i_{n+1},$$

$$i_{n+3}i_n = i_{n+1} = -i_n i_{n+3}.$$

These algorithms are too complicated for a concrete interpretation.

Recently, Graves' octonion has been applied to the study of fundamental particles. However, because the calculations are complicated and octonion mathematics operates in flat space-time, Graves' octonion cannot yield precise conclusions on the fundamental nature of particles. On the contrary, because the new octonion performs simple operations in curved space–time, it may become a powerful tool for fundamental particle study.

11.3 The new octonion

Unlike Graves' octonion, the algorithms of the new quaternion, i.e., the new octonion, are obtained by adding

$$h^2 = -1,\ hi = ih,\ hj = jh,\ hk = kh$$

to the algorithms of Hamilton's quaternion, i.e.,

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

Thus, the new octonion is easily interpretable. In addition, the properties of curved four-dimensional space–time, to date explained by special relativity, can be mathematically demonstrated by the new octonion. Furthermore, whereas the seven imaginary numbers square to -1 in Graves' octonion, in the new octonion, h, i, j, and k square to -1 while the square of real number a and the multiples hi, hj, and hk become $+1$. This difference originates from the fact that the new octonion operates in curved space–time.

Graves' octonion is difficult to understand because it operates in flat space–time, whereas real space–time is curved. When curved space–time phenomena are calculated by the standard octonion, the results of the calculation may become complicated. On the other hand, calculations are easily performed in curved space–time using the new octonion. That is, the new octonion underlies the mathematics of real-world phenomena.

Next, we consider magnitude $|A|$ of a new octonion A, where A is described as

$$A = ah + bi + cj + dk + p + qhi + rhj + shk.$$

This number contains real components a, b, c, d, p, q, r, and s and imaginary components h, i, j, and k. From the definition of the complex conjugate, new complex conjugate, quaternion conjugate, and new quaternion conjugate, we define a new octonion conjugate as

$$\overline{A} = ah - bi - cj - dk + p - qhi - rhj - shk,$$

where all numbers other than imaginary number ah and real number p are negative. In addition, if the the squares of complex magnitudes $|A|^2 = A\overline{A}$ are also realized by the new octonion, we can write

$$\begin{aligned} |A|^2 &= A\overline{A} \\ &= (ah + bi + cj + dk + p + qhi + rhj + shk) \\ &\quad \times (ah - bi - cj - dk + p - qhi - rhj - shk). \end{aligned} \tag{11.1}$$

By analogy to the complex number, new complex number, quaternion, and new quaternion, the reader might consider that (11.1) equates to

$$\begin{aligned} |A|^2 &= (ah)^2 - (bi)^2 - (cj)^2 - (dk)^2 + p^2 - (qhi)^2 - (rhj)^2 - (shk)^2 \\ &= -a^2 + b^2 + c^2 + d^2 + p^2 - q^2 - r^2 - s^2. \end{aligned} \tag{11.2}$$

However, this is not the case.

Instead, we obtain

$$\begin{aligned} |A|^2 &= A\overline{A} \\ &= (ah + bi + cj + dk + p + qhi + rhj + shk) \\ &\quad \times (ah - bi - cj - dk + p - qhi - rhj - shk) \\ &= [(ah + p) + (b + qh)i + (c + rh)j + (d + sh)k] \\ &\quad \times [(ah + p) - (b + qh)i - (c + rh)j - (d + sh)k] \\ &= (ah + p)^2 - (b + qh)^2 i^2 - (c + rh)^2 j^2 - (d + sh)^2 k^2 \\ &= (ah + p)^2 + (b + qh)^2 + (c + rh)^2 + (d + sh)^2. \end{aligned} \tag{11.3}$$

The difference between (11.3) and (11.2) is

$$2aph + 2bqh + 2crh + 2dsh.$$

Thus, $|A|^2$ does not equate to (11.2).

11.4 Double structure of four-dimensional space-time

To interpret (11.3), we can state that the new octonion expresses a four-dimensional space–time formed by four coordinate axes, namely, the $(ah+p)$-, $(b+qh)i$-, $(c+rh)j$-, and $(d + sh)k$-axes. Alternatively, each axis of four-dimensional space-time is a complex number composed of a real and an imaginary number. To ensure that new quaternion calculations are consistent with special relativity, the $(ah + p)$-axis is assigned as the temporal axis while the $(b + qh)i$-, $(c + rh)j$-, and $(d + sh)k$-axes become space axes.

If p is multiplied by imaginary number h and the $(ah+p)$-axis is transformed into the $(a+ph)$-axis, the equation is simplified apparently. However, this transformation restores Hamilton quaternion $a + bi + cj + dk$, which no longer operates in curved space-time. In this book, we have sequentially advanced from the new complex number to new octonion

$$ah + bi + cj + dk + p + qhi + rhj + shk.$$

When developing the mathematics of four-dimensional space–time, the most esthetically pleasing octonion is

$$a + bi + cj + dk + ph + qhi + rhj + shk,$$

which leads us astray from the new octonion.

The mathematical conclusion that each coordinate axis of the four-dimensional space–time is a complex number may be difficult to accept per se, but one interpretation is readily acceptable. Four-dimensional space–time can be regarded as two overlapping space–times. The coordinates of our own space–time are $(ah,\ bi,\ cj,\ dk)$, and those of alternative space–time are $(p,\ qhi,\ rhj,\ shk)$, which we cannot observe. At the same time, the inhabitants of the alternative coordinate system $(p,\ qhi,\ rhj,\ shk)$ cannot observe the coordinates $(ah,\ bi,\ cj,\ dk)$ of our space–time. We call this interpretation the double structure of four-dimensional space–time. In the future, reseachers may accept that each coordinate axis of four-dimensional space–time is a complex number. Currently, we assume more intuitive double structure of four-dimensional space-time. That is, we consider that each coordinate axis of two four-dimensional space–times overlaps and coexists. In simple terms, we regard our world and the alternative world as the positive and negative world, respectively. Some researchers advocate the parallel world scenario in which many universes exist side-by-side. Mathematically, we can demonstrate that two four-dimensional space–times overlap; equivalently, that four-dimensional space–time possesses four complex axes.

At this point, readers may be wondering why the octonion, as an eight-element number, does not represent an eight-dimensional space–time. However, as indicated in (11.3), the new octonion expresses a four-dimensional space–time, whose coordinate axes are complex numbers. Thus, the new octonion expresses two overlapped four-dimensional space–times. Because the new octonion is the highest number class permitted by the algebraic theorem, a larger dimensional space–time more than four-dimensions cannot exist. While higher dimensions are frequently adopted in cosmology, such as sixteen-dimensional cosmology, these cosmologies are inconsistent with the algebraic theorem.

The world of imaginary numbers is often regarded as the imaginary world. However, the new octonion permits a world in which four coordinate axes are expressed by the imaginary numbers ah, bi, cj, and dk while the real numbers p, qhi, rhj, and shk are assigned to the cooordinates of the negative world. In the previous discussion, we arbitrarily referred to our world as the positive world. Strictly speaking, our world and the alternative world are worlds of imaginary and real numbers, respectively.

11.5 New octonions and new Lorentz transformations in the negative world

In this section, we investigate the structure of the negative world in terms of the double structure theory of four-dimensional space–time. The coordinates of the positive world, in which we live, are $(cth,\ xi,\ yj,\ zk)$ and a world point is represented by new octonion $cth + xi + yj + zk$. Therefore, in new octonion

$$A = ah + bi + cj + dk + p + qhi + rhj + shk,$$

the variables ah, bi, cj, and dk are equivalent to cth, xi, yj, and zk, respectively. Then, what is the physical meaning of the remaining variables p, qhi, rhj, and shk?

Multiplying the new octonion $cth + xi + yj + zk$, specifying a world point in the positive world, by h, we obtain $-ct + xhi + yhj + zhk$, which contains the same terms as the remaining variables p, qhi, rhj, and shk of the new octonion. Thus, it is mathematically natural to replace p, qhi, rhj, and shk by $-ct$, xhi, yhj, and zhk. The new octonion $-ct + xhi + yhj + zhk$ may specify a world point in the negative world. Replacing $-ct$ of this new octonion with ct, a world point in the negative world can also be designated as $ct + xhi + yhj + zhk$. The correct form is established by investigating the Lorentz transformations in the negative world, assuming that each new octonion is proper.

(1) If the new octonion in the negative worlds is $-ct + xhi + yhj + zhk$

We replace the new octonion $cth + xi + yj + zk$ describing a world point in the positive world with a new octonion $-ct + xhi + yhj + zhk$ describing a world point in the negative world, and apply the calculations in Section 10.1 to obtain the Lorentz transformations. We assume that motion occurs along the x-axis; that is, $y = z = 0$. In the negative world, we assume that observer A is at rest at the origin O, while observer B moves alomg the x-axis at constant velocity v. In addition, a fixed observed point D lies at distance x, and observers A and B coincide at O at time $t = 0$. After t seconds, A remains at the origin, while B has traveled distance vt. D remains fixed at distance x. The coordinates of the three objects are $A(-ct,\ 0)$, $B(-ct,\ vthi)$, and $D(-ct,\ xhi)$, and their new octonions are $A = -ct,\ B = -ct + vthi$, and $D = -ct + xhi$. Observer A notes that time t has passed but his distance x remains 0. Thus, A moves along the $-ct$-axis. Observer B moves along the straight line $x = vt$. A space–time diagram of this situation will be illustrated later.

The transformation formula $D\overline{B}/\,|B|$ expresses the coordinate transformation of D observed by B. In other words, it expresses the coordinates of D from the reference frame of B, moving along a straight line with uniform velocity. In terms of the new octonions, $D\overline{B}/\,|B|$ is calculated using the new octonions $B = -ct+vthi$ and $D = -ct + xhi$ to obtain

$$\begin{aligned}\frac{D\overline{B}}{|B|} &= \frac{(-ct + xhi)(-ct - vthi)}{\sqrt{(-ct + vthi)(-ct - vthi)}} \\ &= \frac{c^2t^2 + cvt^2hi - xcthi - xvth^2i^2}{\sqrt{(-ct)^2 - (vthi)^2}} \\ &= \frac{c^2t^2 - xvt + cvt^2hi - xcthi}{\sqrt{(-ct)^2 - (vthi)^2}}. \end{aligned} \tag{11.4}$$

Before removing ct from the square root in the denominator of (11.4), we investigate its sign. If $ct = x = y = z = 1$ in the new octonion $cth+xi+yj+zk$ describing a world point in the positive world, the new octonion becomes $h+i+j+k$, the basic new octonion in the positive world. Multiplying the new octonion $cth+xi+yj+zk$ by h, we obtain the new octonion $-ct+xhi+yhj+zhk$, describing a world point in the negative world. Thus, the basic new octonion $-1+hi+hj+hk$ of the negative world can be obtained by multiplying $h+i+j+k$ by h. From this result, the time component of the negative world seems to be negative; that is, $-ct < 0$, or $ct > 0$.

Since

$$(hi)^2 = h^2i^2 = (-1)(-1) = 1,$$

removing ct from the square root of the denominator of (11.4) under the condition $ct > 0$, we obtain

$$\begin{aligned}\frac{D\overline{B}}{|B|} &= \frac{c^2t^2 - xvt + cvt^2hi - xcthi}{\sqrt{(-ct)^2 - (vthi)^2}} \\ &= \frac{c^2t^2 - xvt + cvt^2hi - xcthi}{ct\sqrt{1 - (vthi)^2/(-ct)^2}} \\ &= \frac{c^2t(t - vx/c^2) - ct(x - vt)hi}{ct\sqrt{1 - v^2/c^2}} \\ &= \frac{c(t - vx/c^2) - (x - vt)hi}{\sqrt{1 - v^2/c^2}}. \end{aligned} \tag{11.5}$$

Since (11.5) describes the coordinates $-ct' + x'hi$ of D as seen from B, we have

$$-ct' + x'hi = \frac{c(t - vx/c^2) - (x - vt)hi}{\sqrt{1 - v^2/c^2}}.$$

Comparing the coefficients, we can write

$$t' = \frac{-t + (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{11.6}$$

$$x' = \frac{-x + vt}{\sqrt{1 - v^2/c^2}}. \tag{11.7}$$

The signs of (11.6) and (11.7) are contrary to the equations of the Lorentz transformations of special relativity

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{3.3}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}. \tag{3.4}$$

If $x = 0$ is assumed in (11.6), we find that

$$t' = -t \,/\sqrt{1 - v^2/c^2}$$

and t' is of opposite sign to t. In other words, stationary observer A and moving observer B observe D at times of oppsosite sign. Thus, if we assume that the new octonion of a world point in the negative world is $-ct + xhi + yhj + zhk$, a contradiction occurs. Therefore, we exclude the possibility of $-ct+xhi+yhj+zhk$ and $ct > 0$ in the negative world.

(2) If the new octonion of the negative world is $ct + xhi + yhj + zhk$

In terms of the new octonions $B = ct + vthi$ and $D = ct + xhi$, $D\overline{B}/|B|$ becomes

$$\begin{aligned}\frac{D\overline{B}}{|B|} &= \frac{(ct + xhi)(ct - vthi)}{\sqrt{(ct + vthi)(ct - vthi)}} \\ &= \frac{c^2t^2 - cvt^2hi + xcthi - xvth^2i^2}{\sqrt{(ct)^2 - (vthi)^2}} \\ &= \frac{c^2t^2 - xvt - cvt^2hi + xcthi}{\sqrt{(ct)^2 - (vt)^2}}.\end{aligned} \tag{11.8}$$

Before removing ct from the square root of the denominator of (11.8), we investigate its sign. In this case, $-ct$ is replaced by ct in the new octonion $-ct + xhi + yhj + zhk$. Replacing -1 in the basic new octonion $-1 + hi + hj + hk$ of case (1) with 1, the basic new octonion in the negative world becomes $1 + hi + hj + hk$. In this scenario, the time component ct of the new octonion $ct + xhi + yhj + zhk$ is positive, i.e., $ct > 0$. Therefore, removing ct from the square root of the denominator of (11.8), we obtain

$$\begin{aligned}\frac{D\overline{B}}{|B|} &= \frac{c^2t^2 - xvt - cvt^2hi + xcthi}{\sqrt{(ct)^2 - (vt)^2}} \\ &= \frac{c^2t(t - vx/c^2) + ct(x - vt)hi}{ct\sqrt{1 - v^2/c^2}} \\ &= \frac{c(t - vx/c^2) + (x - vt)hi}{\sqrt{1 - v^2/c^2}}.\end{aligned} \tag{11.9}$$

The new octonion, expressing the coordinates of D as seen from B, is $ct' + x'hi$. Thus, from (11.9), we have

$$ct' + x'hi = \frac{c(t - vx/c^2) + (x - vt)hi}{\sqrt{1 - v^2/c^2}}.$$

Comparing the coefficients, we can write

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{11.10}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}. \tag{11.11}$$

Equations (11.10) and (11.11) are exactly the equations of the Lorentz transformations of special relativity

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{3.3}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}. \tag{3.4}$$

In addition, if $x = 0$, (11.10) reduces to

$$t' = t/\sqrt{1 - v^2/c^2}.$$

Thus, unlike case (1), time does not change sign when observed by different observers. The new octonion $ct + xhi + yhj + zhk$ with $ct > 0$ appropriately represents a world point in the negative world.

Before constructing the four-dimensional space–time diagram in case (2), we elucidate the meaning of $ct > 0$. Because $y = z = 0$, the diagram reduces to a new complex plane. The expression $ct > 0$ permits two cases: if $c > 0$, we have $t > 0$, and if $c < 0$, we have $t < 0$. Because y' and z' in the new positive-world Lorentz transformations involve h, then y' and z' reside in the negative world. If the velocity of light c changes sign in the positive or negative world, the new Lorentz transformations will not be realized. Thus, we may appropriately consider $c > 0$ and $t > 0$ in both positive and negative worlds. When $c > 0$, we have $v > 0$. When $v > 0$ and $t > 0$, we have $x > 0$. Thus, the world line $x = vt$ of observer B in the negative world is a straight line progressing in the upper-right direction from the origin O in the new complex plane. The world line of light similarly progresses as a straight line in the upper-right direction, but remains equidistant from the ct- and xhi-axes. Both world lines are illustrated in Figure 11.1. If this diagram is compared to the new complex plane in the positive world, the reader will appreciate that the positive and negative worlds overlap.

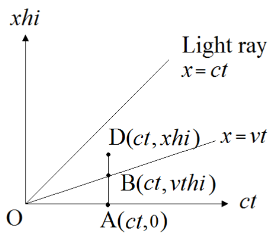


Figure 11.1

From (11.10) and (11.11), we draw an important conclusion that the same physical law, described by the same formula, applies in both positive and negative worlds. Consequently, it is thought that matter and gravity exist in both worlds. This finding is important to fundamental physics.

11.6 World distance in the negative world

We calculate the world distance s given that the new octonion of a world point A in the negative world is $A = ct + xhi + yhj + zhk$. The algorithms are as follows:

$$h^2 = i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,$$

$$hi = ih,\ hj = jh,\ hk = kh.$$

Thus, we find

$$\begin{aligned} s^2 &= A\overline{A} \\ &= (ct + xhi + yhj + zhk)(ct - xhi - yhj - zhk) \\ &= (ct)^2 - ctxhi - ctyhj - ctzhk \\ &\quad + xcthi - (xhi)^2 - xyh^2ij - xzh^2ik \\ &\quad + ycthj - yxh^2ji - (yhj)^2 - yzh^2jk \\ &\quad + ctzhk - zxh^2ki - zyh^2kj - (zhk)^2 \\ &= c^2t^2 - ctxhi - ctyhj - ctzhk + ctxhi - x^2 + xyk - zxj \\ &\quad + ctyhj - xyk - y^2 + yzi + ctzhk + zxj - yzi - z^2 \\ &= c^2t^2 - x^2 - y^2 - z^2. \end{aligned} \tag{11.12}$$

Reconsider that, in Section 7.2, we derived the world distance in the positive world as

$$s^2 = -c^2t^2 + x^2 + y^2 + z^2. \tag{7.6}$$

The sole difference between (11.12) and (7.6) is reversal of sign. Furthermore, if $c > v$, the square of the world distance in the positive and negative worlds, is negative and positive, respectively.

According to this result, our world is strictly the negative world while the so-called negative world is the positive world. However, since this concept is unacceptable to many individuals, we refer our world as the positive world until the new octonion gains wider acceptance.

In addition, the world distance derived from special relativity (see Section 7.1)

$$s^2 = (ct)^2 - x^2 - y^2 - z^2 \tag{7.1}$$

expresses the world distance in the negative world.

11.7 World point in double-structured four-dimensional space-time

In the following discussions, we assume that the new octonion representing the world point in the negative world is $ct + xhi + yhj + zhk$. Thus, the new octonion of a world point in double structured four-dimensional space–time is

$$A = ct_0h + x_0i + y_0j + z_0k + ct_1 + x_1hi + y_1hj + z_1hk. \tag{11.13}$$

Equation (11.13) can be written as

$$A = (ct_0h + ct_1) + (x_0 + x_1h)i + (y_0 + y_1h)j + (z_0 + z_1h)k.$$

Assumed conditions are $c > 0,\ t_0 > 0,\ t_1 > 0$ and the algorithms are as follows:

$$h^2 = i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,$$

$$hi = ih,\ hj = jh,\ hk = kh.$$

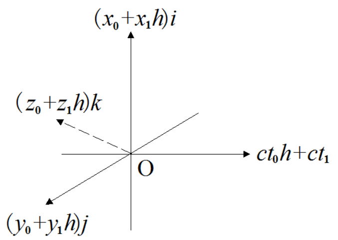


Figure 11.2

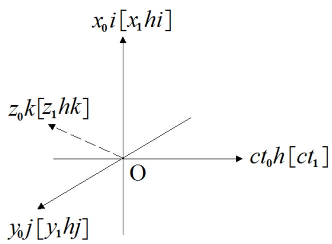


Figure 11.3

Figure 11.2 is a four-dimensional space–time diagram in which each coordinate axis is specified by a complex number. The four-dimensional space–time diagram in Figure 11.3 assumes that four-dimensional space–time has a double structure. Because the zk- and zhk-axes cannot be projected onto the diagram, they are indicated by dashed lines. In addition, when the positive and negative worlds are synchronously illustrated, we denote the coordinate axes in the negative world by $[ct]$, $[xhi]$, $[yhj]$, and $[hkz]$. Even if the coordinate elements include components of both worlds, confusion is prevented by enclosing imaginary numbers in parenthesis, e.g., $(cth,\ xi,\ yj,\ zhk)$ in Chapter 6. The coordinates in the positive and negative worlds are different ($(ah,\ bi)$ and $(a,\ bhi)$, respectively). In contrast, in Hamilton's notation, real numbers are expressed in $(a,\ b)$ format, allowing no distinction between positive and negative worlds. Thus, Hamilton's notation is inconvenient for our purpose.

Describing a world point in the whole four-dimensional space–time by (11.13), can we determine whether t_0 and t_1 are identical? Similarly, are x_0 and x_1, y_0 and y_1, and z_0 and z_1 identical? To answer this question, we rewrite (11.13) as

$$\begin{aligned} B &= cth + xi + yj + zk + ct + xhi + yhj + zhk \\ &= (cth + ct) + (xi + xhi) + (yj + yhj) + (zk + zhk), \end{aligned} \tag{11.14}$$

and investigate whether the coordinate transformation $D\overline{B}/|B|$ in terms of (11.14) yields the Lorentz transformations.

The new octonion of observer B moving with uniform velocity v in the positive x-direction away from stationary observer A is

$$\begin{aligned} B &= (cth + ct) + (vti + vthi) \\ &= ct(1+h) + vti(1+h) \\ &= (1+h)(ct + vti). \end{aligned}$$

The new octonion conjugate is

$$\begin{aligned} \overline{B} &= (cth + ct) - (vti + vthi) \\ &= (1+h)(ct - vti) \end{aligned}$$

and the new octonion of observed point mass D is

$$\begin{aligned} D &= (cth + ct) + (xi + xhi) \\ &= ct(1+h) + xi(1+h) \\ &= (1+h)(ct + xi). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \frac{D\overline{B}}{|B|} &= \frac{(1+h)(ct+xi)(1+h)(ct-vti)}{\sqrt{(1+h)(ct+vti)(1+h)(ct-vti)}} \\ &= \frac{(1+h)(ct+xi)(ct-vti)}{\sqrt{(ct+vti)(ct-vti)}} \\ &= \frac{(1+h)(c^2t^2 - cvt^2i + xcti - xvti^2)}{\sqrt{c^2t^2 - v^2t^2i^2}} \\ &= \frac{(1+h)(c^2t^2 + xvt - cvt^2i + xcti)}{\sqrt{c^2t^2 + v^2t^2}} \\ &= \frac{(1+h)\left[c^2t(t + vx/c^2) + ct(x - vt)i\right]}{ct\sqrt{1 + v^2/c^2}} \end{aligned}$$

$$= \frac{(1+h)\left[c(t+vx/c^2)+(x-vt)i\right]}{\sqrt{1+v^2/c^2}}$$
$$= \frac{c(t+vx/c^2)}{\sqrt{1+v^2/c^2}}h + \frac{x-vt}{\sqrt{1+v^2/c^2}}i + \frac{c(t+vx/c^2)}{\sqrt{1+v^2/c^2}} + \frac{x-vt}{\sqrt{1+v^2/c^2}}hi.$$

This expression is the new octonion $D' = ct'h + x'i + ct' + x'hi$ of D observed from the reference frame of B. Clearly, this expression is inconsistent with the Lorentz transformations. Therefore, the positive and negative worlds cannot share common t, x, y, and z, as assumed in (11.14). In addition, because c^2 of $\sqrt{1+v^2/c^2}$ is non-negative, (11.14) is thought to revert the new octonion to the mathematics of flat space–time.

11.8 Four real numbers and four imaginary numbers in four-dimensional space-time

In this section, we expand the concepts introduced in Section 11.1. More specifically, we investigate whether the hi, hj, and hk components of a new octonion $ah + bi + cj + dk + p + qhi + rhj + shk$ are imaginary or real.

Real and imaginary numbers are defined by the sign of their squares; a squared real number is positive, while a squared imaginary number is negative. The new octonion algorithm gives

$$h^2 = i^2 = j^2 = k^2 = -1$$

implying that h, i, j, and k are imaginary. On the other hand, since

$$hi = ih, \ hj = jh, \ hk = kh,$$

we have

$$\begin{aligned}(hi)^2 &= hihi \\ &= h^2 i^2 \\ &= (-1)(-1) \\ &= 1, \\ (hj)^2 &= 1, \\ (hk)^2 &= 1.\end{aligned}$$

Thus, hi, hj, and hk are considered to be real numbers. In contrast, squaring ij, jk, and ki, we get

$$(ij)(ij) = k^2 = -1,$$
$$(jk)(jk) = i^2 = -1,$$
$$(ki)(ki) = j^2 = -1.$$

Therefore, ij, jk, and ki are imaginary numbers and are distinct from their real counterparts hi, hj, and hk.

As proven in Section 11.5, the new octonion of a world point in the positive world, in which we live, is $cth + xi + yj + zk$, while the negative world equivalent is $ct + xhi + yhj + zhk$. Thus, h is a time-imaginary number and i, j, k are space-imaginary numbers. Similarly, the real numbers a used in our daily routines are time-real numbers, and hi, hj, and hk are space-real numbers. In addition, all coordinate axes in the positive and negative worlds are represented by imaginary and real numbers, respectively. The real number a is also the time-real number in the negative world. In addition, a can be called a scale-real number because it scales as $a \times i = ai$.

At present, while multiple imaginary numbers are recognized, namely, i, j, and k in Hamilton's quaternion

$$a + bi + cj + dk$$

and i_1, i_2, i_3, i_4, i_5, i_6, and i_7 in Graves' octonion

$$a + b_1i_1 + b_2i_2 + b_3i_3 + b_4i_4 + b_5i_5 + b_6i_6 + b_7i_7,$$

it appears that a single real number exists. However, as mentioned above, we can define three space-real numbers hi, hj, and hk in addition to the scalable time-real number. If this is true, only the scale real number and the imaginary number i have been adopted in mathematics and physics, although four real and four imaginary numbers exist in four-dimensional space–time. Quantum mechanics has exploited only the imaginary number i, while Hamilton's quaternion has been largely unused.

Given the rich content of the new octonion, we may naturally consider that, unless all eight numbers are adopted, phenomena in four-dimensional space–time will never be understood completely. The eight numbers contained in the new octonions allow precise calculation of four-dimensional space–time phenomena. Supporting this conclusion, coordinate transformation $\overline{A}/|A|$, which does not consider light, retrieves the equations of special relativity

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{3.3}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \tag{3.4}$$

plus two new equations

$$y' = \frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}}, \tag{10.5}$$

$$z' = \frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}}, \tag{10.6}$$

which have been overlooked in special relativity. Equations (10.5) and (10.6) emerged only after applying the new octonion.

12

Oblique Coordinates of Four-Dimensional Space-Time

12.1 Oblique coordinates of motion along x-axis

To verify the correctness of a coordinate transformation, we must investigate how coordinate axes change after a coordinate transformation. In Chapter 4, it was proven that, after a coordinate transformation, coordinate axes become oblique coordinate axes in two-dimensional space–time. In this chapter, we investigate the relationship between the cth-, xi-, yj-, and zk-axes of observer A and the $ct'h$-, $x'i$-, $y'j$-, and $z'k$-axes of observer B in four-dimensional space–time.

Conclusions obtained in this chapter may be unnecessary for the reader whose specialty is neither physics nor mathematics. However, the following conclusions are important: A coordinate transformation of the yj- and zk-axes transforms them into the $y'j$-axis in the positive world and the $z'hk$-axis in the negative world. Therefore, the yj-zk plane becomes the $y'j$-$[z'hk]$ plane, which contains each coordinate axis in the positive and negative worlds. In other words, the positive and negative worlds do not exist independently but are mixed like a mosaic. These conclusions are proven below.

It is assumed that, for observer A, observer B moves in linear uniform motion with velocity v in the x-direction. The four-dimensional space–time diagram for this motion is shown in Figure 12.1. The thick solid lines are the cth-, xi-, and yj-axes, respectively, and the thick dashed line is the zk-axis for observer A. The thin full lines are the $ct'h$-, $x'i$-, and $y'j$-axes, respectively, and the thin dashed line is the $z'k$-axis for observer B. In addition, the $ct'h$-axis is the world line $x = vt$ of observer B, and both the y and z coordinates are zero for B.

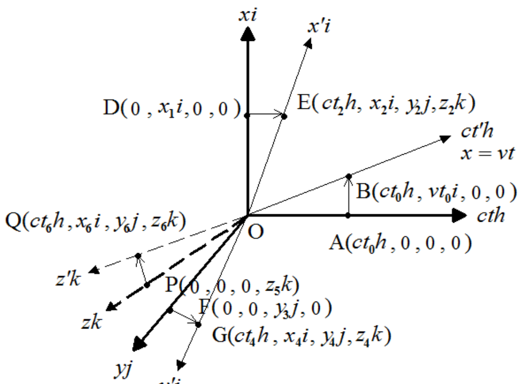


Figure 12.1

Since the time ct_0 for A and B observed by A is the same, the point $A(ct_0h,\ 0,\ 0,\ 0)$ on the cth-axis moves to the point $B(ct_0h,\ vt_0i,\ 0,\ 0)$ on the $ct'h$-axis by a coordinate transformation that changes the cth-axis into the $ct'h$-axis. In addition, it is assumed that the point $D(0,\ x_1i,\ 0,\ 0)$ on the xi-axis moves to the point $E(ct_2h,\ x_2i,\ y_2j,\ z_2k)$ on the $x'i$-axis. The point $F(0,\ 0,\ y_3j,\ 0)$ on the yj-axis moves to the point $G(ct_4h,\ x_4i,\ y_4j,\ z_4k)$ on the $y'j$-axis. Similarly, the point $P(0,\ 0,\ 0,\ z_5k)$ on the zk-axis moves to the point $Q(ct_6h,\ x_6i,\ y_6j,\ z_6k)$ on the $z'k$-axis.

Because the migration from A to B constitutes a rotation about the origin, it can be denoted by a new octonion H. Since the new octonions of point A and point B are $A = ct_0h$ and $B = ct_0h + vt_0i$, respectively, from $AH = B$, we have

$$\begin{aligned} ct_0hH &= ct_0h + vt_0i, \\ H &= \frac{ct_0h + vt_0i}{ct_0h} \qquad (\because\ t_0 \neq 0) \\ &= 1 + \frac{vi}{ch} \\ &= 1 + \frac{vih}{ch^2} \\ &= 1 - \frac{v}{c}hi. \end{aligned} \tag{12.1}$$

Equation (12.1) transforms the cth-axis into the $ct'h$-axis.

The new octonions of D and E are $D = x_1i$ and $E = ct_2h + x_2i + y_2j + z_2k$, respectively. Since D is moved to E by the transformation H, from $DH = E$, we can write

$$x_1iH = ct_2h + x_2i + y_2j + z_2k.$$

Substituting (12.1) into this equation gives

$$\begin{aligned} ct_2h + x_2i + y_2j + z_2k &= x_1i(1 - \frac{v}{c}hi) \\ &= x_1i - \frac{vx_1}{c}hi^2 \\ &= x_1i + \frac{v}{c}x_1h \\ &= \frac{v}{c}x_1h + x_1i. \end{aligned}$$

By comparing the coefficients, we have

$$ct_2 = \frac{v}{c}x_1,\ x_2 = x_1,\ y_2 = 0,\ z_2 = 0.$$

Eliminating x_1 from these equations, we find

$$x_2 = \frac{c}{v}(ct_2),\ y_2 = z_2 = 0. \tag{12.2}$$

(12.2) describes the $x'i$-axis. Because the unit of the temporal axis is ct, the gradient of the $x'i$-axis is not c^2/v but c/v. Because the gradient of the $ct'h$-axis, which is the result of the transformation of the cth-axis, is v/c from $x = vt = (v/c)ct$, the $x'i$-axis and the $ct'h$-axis lean inward at the same angle with respect to the xi- and cth-axes, respectively; i.e., the $x'i$- and $ct'h$-axes are oblique coordinate axes. However, since $y_2 = z_2 = 0$, oblique coordinate axes lie in the two-dimensional cth-xi plane.

Next, the y'-axis is obtained from the y-axis. The new octonions of F and G are $F = y_3j$ and $G = ct_4h + x_4i + y_4j + z_4k$, respectively. From $FH = G$, we can write

$$y_3jH = ct_4h + x_4i + y_4j + z_4k.$$

Substituting (12.1) into this equation gives

$$\begin{aligned} ct_4h + x_4i + y_4j + z_4k &= y_3j(1 - \frac{v}{c}hi) \\ &= y_3j - \frac{v}{c}y_3hji \\ &= y_3j + \frac{v}{c}y_3hk. \end{aligned}$$

By comparing the coefficients, we find

$$ct_4 = 0,\ x_4 = 0,\ y_4 = y_3,\ z_4 = \frac{v}{c}y_3h.$$

If y_3 is eliminated from these equations, we are left with

$$z_4 = \frac{v}{c}y_4h,\ ct_4 = x_4 = 0. \tag{12.3}$$

(12.3) describes the $y'j$-axis. If this is drawn on the yj-$[zhk]$ plane with the condition $ct = x = 0$, we obtain the result shown in Figure 12.2. Since z has h, it becomes the $[zhk]$-axial ingredient in the negative world.

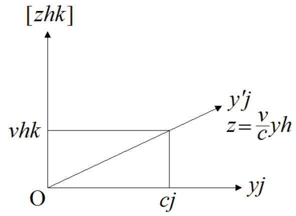


Figure 12.2

As explained in Section 11.7, the zk-axis is the coordinate axis in the positive world and the $[zhk]$-axis is the coordinate axis in the negative world. Thus, as shown in Figure 12.2, the horizontal axis is the yj-axis in the positive world, and the longitudinal axis is the $[zhk]$-axis in the negative world. From this figure, we see that the positive and negative worlds do not exist independently but are mixed like a mosaic. To better understand, note that one four-dimensional space–time, where coordinate components are complex numbers, has tentatively been divided into a positive world and a negative world. Thus, it is not surprising that the coordinate axes in the positive and negative worlds are intermingled. Strictly speaking, one four-dimensional space–time with complex coordinates exists and the axes of the positive and negative world are observed as dictated by conditions.

Here, we reconfirm that the positive world and the negative world are incorrect terms—rather, we should speak of a world of imaginary numbers and a world of real numbers. The terms positive world and negative world are used to make things easier to understand. However, these terms should not be taken to imply that all numbers are positive in the positive world and negative in the negative world. These terms are fallbacks because the more appropriate terminology, i.e., the world and antiworld, was already used in the physics vocabulary, i.e., matter and antimatter, and so could not be adopted. When the contents of this book become more widely known, the terms positive world and negative world should be changed to the world of imaginary numbers and the world of real numbers.

Because, from (12.3), the gradient of the $y'j$-axis with respect to the yj-axis is v/c, the slope of the $y'j$-axis is v/c with respect to the original coordinate axis as well as the $ct'h$- and $x'i$-axes. Therefore, the $y'j$-axis becomes an oblique coordinate axis.

We now perform the same calculations for the $z'k$-axis. The point $P(0,\ 0,\ 0,\ z_5k)$

on the zk-axis transforms to the point $Q(ct_6h,\ x_6i,\ y_6j,\ z_6k)$ on the $z'k$-axis. Because their new octonions are $P = z_5k$ and $Q = ct_6h+x_6i+y_6j+z_6k$, respectively, we find the following from $PH = Q$:

$$z_5kH = ct_6h + x_6i + y_6j + z_6k.$$

If (12.1) is substituted into this equation, we find

$$\begin{aligned} ct_6h + x_6i + y_6j + z_6k &= z_5k(1 - \frac{v}{c}hi) \\ &= z_5k - \frac{v}{c}z_5hki \\ &= -\frac{v}{c}z_5hj + z_5k. \end{aligned}$$

Comparing the coefficients, we have

$$t_6 = 0,\ x_6 = 0,\ y_6 = -\frac{v}{c}z_5h,\ z_6 = z_5.$$

If z_5 is eliminated from this formula, we find

$$y_6 = -\frac{v}{c}z_6h,\ t_6 = x_6 = 0. \qquad (12.4)$$

(12.4) describes the $z'k$-axis. As opposed to (12.3), (12.4) refers to the plane that consists of the $[yhj]$-axis in the negative world and the zk-axis in the positive world. If it is illustrated, it becomes as shown in Figure 12.3.

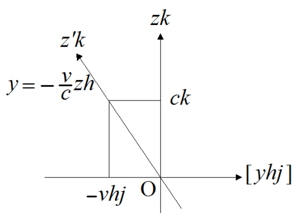


Figure 12.3

Because Figures 12.2 and 12.3 have different coordinate axes, they cannot be stacked. Thus, if both sides of the left equation from (12.4), i.e.,

$$y_6 = -\frac{v}{c}z_6h,$$

are multiplied by ch/v, we have

$$\begin{aligned} \frac{c}{v}y_6h &= -\frac{v}{c}z_6h(\frac{c}{v}h) \\ &= z_6. \end{aligned}$$

Thus, the equation of point Q on the $z'k$-axis becomes

$$z_6 = \frac{c}{v} y_6 h, \ t_6 = x_6 = 0$$

on the yj-$[zhk]$ plane. From the equation of z_6, we find that the $z'k$-axis becomes a straight line whose slope with respect to the $[zhk]$-axis is v/c. Thus, the $z'k$-axis becomes $[z'hk]$-axis because it has h, i.e., the $[z'hk]$-axis forms an oblique coordinate axis with respect to the $y'j$-axis on the yj-$[zhk]$ plane under the conditions $ct = x = 0$ as shown in Figure 12.4.

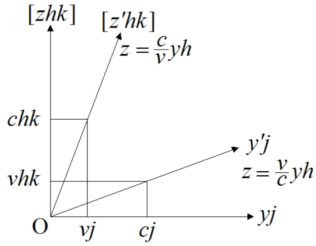


Figure 12.4

The above results prove that the coordinate transformation, that changes the cth-axis into the $ct'h$-axis in four-dimensional space–time, makes four oblique coordinate axes. In addition, when the velocity v of observer B is in the x-direction with respect to observer A, events in four-dimensional space–time can be illustrated and calculated only in two two-dimensional planes: the cth-xi and yj-$[zhk]$ planes. In other words, there is no need to consider a complicated four-dimensional space–time diagram.

The fact that the cth-xi and yj-$[zhk]$ planes are independent can be proven by starting with the equation

$$-c^2t'^2 + x'^2 + y'^2 + z'^2 = -c^2t^2 + x^2 + y^2 + z^2,$$

which shows the invariance of the world distance, which was explained in Section 10.3. This equation can be solved by dividing it into two parts: $-c^2t'^2 + x'^2 = -c^2t^2 + x^2$ and $y'^2 + z'^2 = y^2 + z^2$. The independence of the cth-xi and yj-$[zhk]$ planes follows from the fact that the new and original Lorentz transformations containing ct' and x' do not contain y or z and the equations containing y' and z' do not contain ct or x.

12.2 Oblique coordinates of motion in any direction

When the velocity v of observer B is in the x-direction of observer A, only the cth-xi and the yj-$[zhk]$ planes need be considered. However, when the velocity v of observer B is in an arbitrary direction, we must consider a complicated four-dimensional space–time diagram. The diagram can be understood if the general formula of each coordinate axis after the coordinate transformation, which changes the cth-axis into the $ct'h$-axis, is obtained. In this section, we investigate motion of coordinate axes, in case the velocity v of observer B is in an arbitrary direction. However, because the number of equations increase, only motion of the cth- and xi-axes is calculated.

Assume that the components of the velocity v of observer B as seen by observer A in three-dimensional space are $(v_x i,\ v_y j,\ v_z k)$. Because $v = v_x i + v_y j + v_z k$, we have

$$\begin{aligned} v^2 &= v\bar{v} \\ &= (v_x i + v_y j + v_z k)(-v_x i - v_y j - v_z k) \\ &= v_x^2 + v_y^2 + v_z^2. \end{aligned}$$

The four-dimensional space–time diagram can be obtained by changing the coordinates of point B into $(ct_0 h,\ v_x t_0 i,\ v_y t_0 j,\ v_z t_0 k)$ in Figure 12.1. Since the time ct_0 of A and B is the same, the point $A(ct_0 h,\ 0,\ 0,\ 0)$ on the cth-axis moves to the point $B(ct_0 h,\ v_x t_0 i,\ v_y t_0 j,\ v_z t_0 k)$ on the $ct'h$-axis by the coordinate transformation H. The new octonions of the points A and B are $A = ct_0 h$ and $B = ct_0 h + v_x t_0 i + v_y t_0 j + v_z t_0 k$, respectively, then from $AH = B$, we find

$$\begin{aligned} ct_0 hH &= ct_0 h + v_x t_0 i + v_y t_0 j + v_z t_0 k, \\ H &= \frac{ct_0 h + v_x t_0 i + v_y t_0 j + v_z t_0 k}{ct_0 h} \qquad (\because\ t_0 \neq 0) \\ &= 1 + \frac{1}{ch}(v_x i + v_y j + v_z k) \\ &= 1 + \frac{h}{ch^2}(v_x i + v_y j + v_z k) \\ &= 1 - \frac{h}{c}(v_x i + v_y j + v_z k). \end{aligned} \tag{12.5}$$

Equation (12.5) transforms the cth-axis into the $ct'h$-axis.

We now calculate the gradient of the $ct'h$-axis. The coordinate of the cth-direction of point B is $ct_0 h$. The magnitude of the ingredient $v_x t_0 i + v_y t_0 j + v_z t_0 k$ perpendicular to the cth-axis is

$$\sqrt{(v_x t_0 i + v_y t_0 j + v_z t_0 k)(-v_x t_0 i - v_y t_0 j - v_z t_0 k)}$$
$$= t_0 \sqrt{-v_x^2 i^2 - v_y^2 j^2 - v_z^2 k^2} \ (\because\ t_0 > 0)$$
$$= t_0 \sqrt{v_x^2 + v_y^2 + v_z^2}$$
$$= v t_0.$$

From these calculations, the magnitude of the coordinate in the cth-direction of point B is ct_0, and

$$vt_0/ct_0 = v/c \tag{12.6}$$

is the gradient of the $ct'h$-axis with respect to the cth-axis because the magnitude of the coordinate in the direction perpendicular to the cth-axis is vt_0; i.e., even when the velocity v of B is in an arbitrary direction, the gradient of the $ct'h$-axis after a coordinate transformation is v/c. This result is the same as when the velocity v of B is in the x-direction of A.

For D, the new octonion is $D = x_1 i$. Since D is moved to E by the transformation H, we have

$$E = x_1 i H.$$

If (12.5) is substituted into this equation, we find

$$\begin{aligned} E &= x_1 i \left[1 - \frac{h}{c}(v_x i + v_y j + v_z k)\right] \\ &= x_1 i - \frac{x_1 h}{c}(v_x i^2 + v_y ij + v_z ik) \\ &= x_1 i - \frac{x_1 h}{c}(-v_x + v_y k - v_z j) \\ &= \frac{v_x x_1}{c} h + x_1 i + \frac{v_z x_1}{c} hj - \frac{v_y x_1}{c} hk. \end{aligned} \tag{12.7}$$

(12.7) is the new octonion of the point E on the $x'i$-axis.

We now calculate the gradient of the $x'i$-axis with respect to the xi-axis. The coordinate of point E in the xi-direction is $x_1 i$. With the help of the other coordinates from (12.7), the magnitude of a coordinate in the direction perpendicular to xi-axis is

$$\sqrt{(v_x x_1 h/c + v_z x_1 hj/c - v_y x_1 hk/c)(v_x x_1 h/c - v_z x_1 hj/c + v_y x_1 hk/c)}$$
$$= \sqrt{(v_x x_1 h/c)^2 - (v_z x_1 hj/c)^2 - (v_y x_1 hk/c)^2}$$
$$= (h/c)\sqrt{v_x^2 x_1^2 + v_z^2 x_1^2 + v_y^2 x_1^2}$$
$$= v x_1 h/c. \ (\because\ x_1 > 0)$$

From the above results, the magnitude of the coordinate of point E in the xi-direction is x_1 and the magnitude of the coordinate in the direction perpendicular to the xi-axis is vx_1/c. Thus, the gradient of the $x'i$-axis with respect to the xi-axis is

$$(vx_1/c)/x_1 = v/c. \tag{12.8}$$

Since (12.8) is the same as (12.6), even when the velocity v of B is in an arbitrary direction, the $ct'h$-axis and the $x'i$-axis form oblique coordinate axes.

From the same calculations, the new octonions of point G on the $y'j$-axis and point I on the $z'k$-axis are

$$G = \frac{v_y y_2}{c}h - \frac{v_z y_2}{c}hi + y_2 j + \frac{v_x y_2}{c}hk,$$
$$I = \frac{v_z z_3}{c}h + \frac{v_y z_3}{c}hi - \frac{v_x z_3}{c}hj + z_3 k.$$

The gradients of the $y'j$- and $z'k$-axes can be obtained by using the same methods that led to (12.8). However, this calculation is omitted here because it involves more equations.

12.3 Interpretation of y' and z' using new Lorentz transformations

As obtained in Section 10.1, the new Lorentz transformations in four-dimensional space–time are

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{10.3}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \tag{10.4}$$

$$y' = \frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}}, \tag{10.5}$$

$$z' = \frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}}. \tag{10.6}$$

Although (10.3) and (10.4) are the same as the Lorentz transformations of special relativity, (10.5) and (10.6) differ from the results

$$y' = y, \tag{3.5}$$

$$z' = z \tag{3.6}$$

of the Lorentz transformations. The new transformations include the coordinates hj and hk in the negative world, because (10.5) and (10.6) are

$$y'j = \frac{yj + (v/c)zhj}{\sqrt{1 - v^2/c^2}},$$

$$z'k = \frac{zk - (v/c)yhk}{\sqrt{1 - v^2/c^2}},$$

if written out exactly.

Because the *yj*-*zk* plane changes to the *yj*-[*zhk*] plane after a coordinate transformation, as proven in Section 12.1, (10.5) and (10.6) are considered to contain hj and hk. Thus, we will prove that (10.5) and (10.6) can be obtained from the coordinates of the nodes of straight lines in the *yj*-[*zhk*] plane.

We have proven in Section 12.1 that when the velocity v of observer B is in the x-direction for observer A, events in four-dimensional space–time can be calculated by considering two two-dimensional planes, i.e., the *cth*-*xi* and *yj*-[*zhk*] planes. Thus, we now consider the point $D(y_0j,\ z_0hk)$ on the *yj*-[*zhk*] plane, which is shown in Figure 12.5.

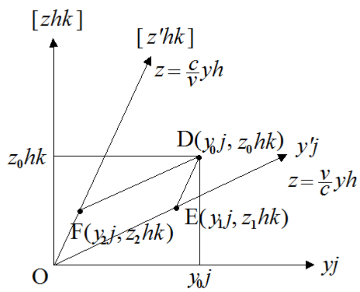


Figure 12.5

The straight lines parallel to the $y'j$- and $[z'hk]$-axes, which are oblique coordinate axes, are drawn from D. They intersect each coordinate axis at $F(y_2j,\ z_2hk)$ and $E(y_1j,\ z_1hk)$. However, if the yj- and zk-axes of stationary observer A are not first replaced by the yj- and $[zhk]$-axes, the calculations do not work. The reason for this is unknown.

(1) New Lorentz transformation formula for y'

The gradient of the straight line DE is c/v, which is the same as the gradient of the $[z'hk]$-axis, and it passes through the point $D(y_0j,\ z_0hk)$. Thus, its equation is

$$z - z_0 = \frac{c}{v}(y - y_0)h. \tag{12.9}$$

In addition, the equation of the $y'j$-axis is

$$z = \frac{v}{c} yh. \tag{12.10}$$

Because (12.9) and (12.10) describe lines that pass through the point $E(y_1 j,\ z_1 hk)$, we have

$$z_1 = \frac{v}{c} y_1 h, \tag{12.11}$$

$$z_1 - z_0 = \frac{c}{v}(y_1 - y_0)h. \tag{12.12}$$

If (12.12) is subtracted from (12.11), we find

$$\begin{aligned}
z_0 &= \frac{v}{c} y_1 h - \frac{c}{v}(y_1 - y_0)h \\
&= (\frac{v}{c} - \frac{c}{v}) y_1 h + \frac{c}{v} y_0 h, \\
(\frac{c}{v} - \frac{v}{c}) y_1 h &= \frac{c}{v} y_0 h - z_0, \\
\frac{c}{v}(1 - v^2/c^2) y_1 h &= \frac{c}{v} y_0 h - z_0, \\
y_1 h &= \frac{1}{1 - v^2/c^2}(y_0 h - \frac{v}{c} z_0), \\
y_1 &= \frac{1}{1 - v^2/c^2}(y_0 + \frac{v}{c} z_0 h).
\end{aligned} \tag{12.13}$$

Substituting (12.13) into (12.11) gives

$$z_1 = \frac{vh/c}{1 - v^2/c^2}(y_0 + \frac{v}{c} z_0 h). \tag{12.14}$$

Let $|OE|$ be the magnitude of the line segment OE, then we have

$$\begin{aligned}
|OE|^2 &= (y_1 j + z_1 k)(-y_1 j - z_1 k) \\
&= -y_1^2 j^2 - z_1^2 k^2 \\
&= y_1^2 + z_1^2.
\end{aligned}$$

If (12.13) and (12.14) are substituted into this equation, the result is

$$\begin{aligned}
|OE|^2 &= \frac{1}{(1 - v^2/c^2)^2}(y_0 + \frac{v}{c} z_0 h)^2 \ + \frac{v^2 h^2/c^2}{(1 - v^2/c^2)^2}(y_0 + \frac{v}{c} z_0 h)^2 \\
&= \frac{1 - v^2/c^2}{(1 - v^2/c^2)^2}(y_0 + \frac{v}{c} z_0 h)^2 \\
&= \frac{1}{1 - v^2/c^2}(y_0 + \frac{v}{c} z_0 h)^2.
\end{aligned}$$

Since $c > v,\ y_0 > 0$, and $z_0 > 0$, we have

$$|OE| = \frac{1}{\sqrt{1 - v^2/c^2}}(y_0 + \frac{v}{c}z_0 h).$$

Since $|OE|$ is the y' coordinate of B, we find

$$y' = \frac{1}{\sqrt{1 - v^2/c^2}}(y_0 + \frac{v}{c}z_0 h).$$

To generalize, y_0 is substituted for y, which gives

$$y' = \frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}}. \tag{12.15}$$

(12.15) is the same as the new Lorentz transformation

$$y' = \frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}}. \tag{10.5}$$

We have obtained it from the coordinate of the node of straight lines in the yj-$[zhk]$ plane.

(2) New Lorentz transformation formula for z'

The new Lorentz transformation formula for z' can be obtained from the coordinate of the node of the straight lines, like (1). However, we obtain the formula by another method. As proven in Section 4.3, in the oblique coordinate system, the formula for the reverse transformation can be obtained by switching y and y' with zh and $z'h$, respectively, in the formula in the original coordinates. If this is done in (12.15), we find

$$z'h = \frac{zh + (v/c)y}{\sqrt{1 - v^2/c^2}}.$$

If both sides of this equation are multiplied by $-h$, we have

$$-hz'h = \frac{-hzh - h(v/c)y}{\sqrt{1 - v^2/c^2}},$$

$$z' = \frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}}. \tag{12.16}$$

(12.16) is the same as the new Lorentz transformation

$$z' = \frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}}. \tag{10.6}$$

Based on (12.15) and (12.16), the coordinate axes after the coordinate transformation in the yj-zk plane are the yj- and $[zhk]$-axes, and the positive world and negative world intermingle. In addition, the same conclusion is obtained even if the coordinate axes are the $[yhj]$-axis in the negative world and the zk-axis in the positive world. However, if the yj- and zk-axes of stationary observer A are not set as the yj- and $[zhk]$-axes at the outset, the calculations do not work for an unknown reason. Instead of thinking that positive and negative four-dimensional space–times overlap, it may be better to envision one four-dimensional space–time, whose coordinate axes are complex numbers.

13

Axioms and Theorems of the New Octonion

13.1 Necessity of axioms and theorems

The new complex number introduced in this book enables a complex representation of the Lorentz transformations. The new numbers have been extended to four-dimensional space–time and the new quaternion, i.e., the new octonion, has been proposed for the first time. To properly advance our theory and ensure accurate future calculations, we must establish axioms and theorems of the new mathematics; in particular, of the new octonion. This is the aim of the current chapter.

First, we briefly discuss the semantic of an axiom and define a theorem. An axiom is a rule that, while unable to be proven, is consistent with fundamental mathematical laws. For instance, $1 + 1 = 2$ is an axiom, since it cannot be proven but is regarded as correct. A theorem is a fundamental law that can be proven using axioms. For instance, $1 + 2 = 3$ is a theorem because it is a composite of two axioms; namely, $1 + 1 = 2$ and that the same quantity can be added to both sides of an equation. A theorem is suspected when its proof is contradictory. When the theorem is incorrect, it is necessary to suspect a single axiom. In other words, all the conclusions of a certain mathematics can ultimately be traced to an axiom. In this sense, an axiom is the fundamental in mathematical unit.

The axiom as the starting point of mathematical discourse was pioneered by the Greeks around 2300 years ago. Among the most famous Greek scholars was Euclid, whose seminal work *Elements* introduces five axioms on geometry and five axioms on the whole mathematics. Correctly speaking, Euclid selected rather than determined his axioms. That is, Euclid's axioms were regarded as correct but could not be proven.

Of Euclid's five axioms of geometry, the most well-known is the fifth axiom, i.e., the fifth postulate, which states that parallel lines do not intersect. Though intuitively correct, endeavors to prove this axiom using the other four axioms led to the birth of non-Euclidean geometry. Parallel lines that do not intersect in Eulciean space can be made to intersect in non-Euclidean, curved-space geometry. Such non-Euclidean geometry, known as Riemannian geometry, was adopted by Einstein in his theory of general relativity. As exemplified by the parallel axiom, once an axiom is determined, other researchers can verify whether the mathematics is correct and can develop new mathematics.

Unlike mathematics, axioms are not defined in physics. In addition, the word axiom does not appear in physics textbooks. However, one premise of Newtonian mechanics, the existence of absolute rest, is an axiom because it is considered correct but cannot be proven. On the other hand, the non-existence of absolute rest is an axiom of special relativity. By merely suspecting an axiom of Newtonian physics, Einstein pioneered a new physics. Therefore, if a physical theory is described by a set of axioms at the outset, contradictory conclusion may be resolvable by suspecting one of these axioms. Here, by establishing the axioms and theorems of the new octonion, any erroneous conclusions identified in this book can be traced to their source. However, since the following axioms and theorems are defined only at this time, they are likely to be rewritten by mathematicians once the new octonion is proven to be correct.

The inclusion of time in the following axioms and theorems may appear incongruous to many readers. However, these concepts present a natural description of the new octonion as a mathematics of four-dimensional space–time. By contrast, Euclid's *Elements* treats three-dimensional space in the absence of any temporal effects.

13.2 Axioms of the new octonion

Axiom 1

The new octonion $\boldsymbol{A}$ is denoted as

$$\boldsymbol{A = ah + bi + cj + dk + p + qhi + rhj + shk,}$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$, and $\boldsymbol{s}$ are real, and $\boldsymbol{h}$, $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are imaginary numbers. The algorithms of $\boldsymbol{A}$ are

$$\boldsymbol{h^2 = i^2 = j^2 = k^2 = -1,}$$

$$\boldsymbol{ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,}$$

$$\boldsymbol{hi = ih,\ hj = jh,\ hk = kh.}$$

In the strictest terms, because $k^2 = -1$ can be obtained from $i^2 = j^2 = -1$ and $ij = -ji = k$, Axiom 1 includes a theorem. Performing the algebra, we find that

$$\begin{aligned} k^2 &= ijij \\ &= i(-ij)j \\ &= -i^2j^2 \\ &= -(-1)(-1) \\ &= -1. \end{aligned}$$

Therefore, $k^2 = -1$ must be deleted from Axiom 1. However, we retain the above definition of Axiom 1 for convenience and ease of understanding.

Axiom 2
The new octonion conjugate $\overline{\boldsymbol{A}}$ of the new octonion $\boldsymbol{A}$ is

$$\boldsymbol{\overline{A} = ah - bi - cj - dk + p - qhi - rhj - shk.}$$

Axiom 3
If the magnitude of the new octonion $\boldsymbol{A}$ is $|\boldsymbol{A}|$, we have

$$\boldsymbol{|A|^2 = A\overline{A}.}$$

Axiom 4
The division $\boldsymbol{A \div B}$ of two new octonions $\boldsymbol{A}$ and $\boldsymbol{B}$ is

$$\boldsymbol{A \div B = \frac{A\overline{B}}{|B|^2}.}$$

Axiom 5
The new octonion $\boldsymbol{B\overline{A}/|A|}$ is the coordinate transformation of $\boldsymbol{B}$ by $\boldsymbol{A}$ in four-dimensional space-time, where $\boldsymbol{A = ah + bi + cj + dk}$ or $\boldsymbol{A = p + qhi + rhj + shk}$.

In Section 2.2, we proved that $B\overline{A}/|A|$ becomes a coordinate transformation in two-dimensional space–time. However, we have not proven whether this transformation is extendible to four-dimensional space–time. We further demonstrated, in Section 11.7, that $B\overline{A}/|A|$ does not become a coordinate transformation when $A = ah + bi + cj + dk + p + qhi + rhj + shk$.

Axiom 6

In the form $\boldsymbol{ct}$ (the product of time $\boldsymbol{t}$ and velocity of light $\boldsymbol{c}$), $\boldsymbol{t}$ can undergo addition and subtraction operations with the spatial dimensions $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$.

Note that since the velocity of light c has unit $[distance]/[time]$, the unit of ct is

$$\frac{[distance]}{[time]} \times [time] = [distance].$$

Axiom 7

The new octonion, describing the world point in four-dimensional space-time, is

$$\boldsymbol{A = ct_0h + x_0i + y_0j + z_0k + ct_1 + x_1hi + y_1hj + z_1hk.}$$

However, as deduced in Section 11.5, $\boldsymbol{ct_0}$ and $\boldsymbol{ct_1}$ are both positive.

That $ct > 0$ is correct in the negative world is an axiom. Moreover, the condition $ct > 0$ implies that $c > 0,\ t > 0$ or $c < 0,\ t < 0$. The discussions advanced in this book have assumed that $c > 0$ in both positive and negative worlds. A consistent sign of c is required to satisfy the y' and z' formulae of the new Lorentz transformations in the negative world.

Axiom 8

The new octonion describing the world point in the positive world (in which we live) is

$$\boldsymbol{ct_0h + x_0i + y_0j + z_0k},\ (\boldsymbol{ct_0 > 0}),$$

while the world point in the synchronously existing negative world is described by

$$\boldsymbol{ct_1 + x_1hi + y_1hj + z_1hk.}\ (\boldsymbol{ct_1 > 0})$$

We can regard four-dimensional space–time as a single entity with complex coordinate axes disregarding its double structure. In this construction, the new octonion describing the world point is

$$A = (ct_0h + ct_1) + (x_0 + x_1h)i + (y_0 + y_1h)j + (z_0 + z_1h)k.$$

Axiom 9

When the quantity inside the square root is negative, the negative sign can be placed outside of the square root.

As explained in Section 3.5, if $c > v$, we have

$$c^2t^2h^2 - v^2t^2i^2 = -c^2t^2 + v^2t^2 < 0.$$

Thus, if $ct > 0$, cth is taken outside of the square root as follows:

$$\begin{aligned}\sqrt{c^2t^2h^2 - v^2t^2i^2} &= cth\sqrt{1 - (v^2t^2i^2)/(c^2t^2h^2)} \\ &= cth\sqrt{1 - v^2/c^2}.\end{aligned}$$

By this manipulation, the quantity in the square root is rendered positive. Since its correctness has yet to be proven, this method constitutes an axiom.

Axiom 10

$$\sqrt{\boldsymbol{h^2}} = \boldsymbol{h}.$$

As shown in Section 3.5, setting $\sqrt{h^2} = -h$ does not recover the Lorentz transformations. Thus, we assume that $\sqrt{h^2} = h$ is correct.

13.3 Theorems of the new octonion

We now prove the theorems of the new octonion. If the content of the theorem denies a certain proposition, the theorem is verified by an example that is not applied to the proposition. In addition, a semantic differs between two-dimensional space–time and two-dimensional space. The former comprises one temporal and one spatial dimension, while the latter comprises two spatial dimensions. The theorems that also hold for real numbers, i.e., $(A+B)+C = A+(B+C)$ and $A(B+C) = AB+AC$, are omitted. Theorems that differ from those of real and complex numbers are included. Most of the illustrated theorems are realized in two dimensions; theorems realized in four dimensions are not shown.

Theorem 1
Given two new octonions $\boldsymbol{A}$ and $\boldsymbol{B}$, we have

$$\boldsymbol{AB \neq BA.}$$

By contrast, two complex numbers $A = a + bi$ and $B = c + di$ are multiplied as

$$\begin{aligned} AB &= (a+bi)(c+di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i, \\ BA &= (c+di)(a+bi) \\ &= ca + cbi + dai + dbi^2 \\ &= (ac - bd) + (ad + bc)i. \end{aligned}$$

Thus, for complex numbers, we have

$$AB = BA.$$

Next, we consider two new octonions A and B. Because the imaginary number k can be obtained from the imaginary numbers i and j, as explained in Axiom 1, it is thought that, without loss of generality, we can omit k when proving the theorems of the new octonions A and B. In addition, since this theorem is a denial, the components of the negative world are also omitted. If two new octonions A and B are written as $A = ah + bi + cj$ and $B = dh + ei + fj$, respectively, we find

$$\begin{aligned} AB &= (ah + bi + cj)(dh + ei + fj) \\ &= adh^2 + aehi + afhj + bdhi + bei^2 + bfij + cdhj + ceji + cfj^2 \\ &= (-ad - be - cf) + (ae + bd)hi + (af + cd)hj + (bf - ce)k, \\ BA &= (dh + ei + fj)(ah + bi + cj) \\ &= dah^2 + dbhi + dchj + eahi + ebi^2 + ecij + fahj + fbji + fcj^2 \\ &= (-ad - be - cf) + (ae + bd)hi + (af + cd)hj - (bf - ce)k. \end{aligned}$$

Comparing AB with BA, the sign of k is reversed. Thus, in the new octonion, we have proven that

$$AB \neq BA.$$

Hamilton's quaternion follows the same law.

Theorem 2

Like complex numbers, a new octonion $\boldsymbol{A}$ and its new octonion conjugate $\overline{\boldsymbol{A}}$ satisfy

$$\boldsymbol{A\overline{A} = \overline{A}A}.$$

Note that although $AB \neq BA$ for two general new octonions A and B (Theorem 1), $A\overline{A} = \overline{A}A$ is realized when a new octonion A is multiplied by its new octonion conjugate $\overline{A}$ (or vice versa). Similar to Theorem 1, we omit the k part of the new octonion in the proof of Theorem 2. First, we consider a new octonion in the positive world.

Given a new octonion $A = ah + bi + cj$ and its conjugate $\overline{A} = ah - bi - cj$, the products are

$$\begin{aligned}
A\overline{A} &= (ah + bi + cj)(ah - bi - cj) \\
&= a^2h^2 - abhi - achj + bahi - b^2i^2 - bcij + cahj - cbji - c^2j^2 \\
&= (-a^2 + b^2 + c^2) - (ab - ba)hi - (ac - ca)hj - (bc - cb)k \\
&= -a^2 + b^2 + c^2, \\
\overline{A}A &= (ah - bi - cj)(ah + bi + cj) \\
&= a^2h^2 + abhi + achj - bahi - b^2i^2 - bcij - cahj - cbji - c^2j^2 \\
&= (-a^2 + b^2 + c^2) + (ab - ba)hi + (ac - ca)hj - (bc - cb)k \\
&= -a^2 + b^2 + c^2,
\end{aligned}$$

whereby we have

$$A\overline{A} = \overline{A}A.$$

The whole new octonion, containing the new octonion in the negative world, can be similarly proved. However, since the calculations are lengthy and more tedious, they are omitted here.

In Axiom 3, the magnitude $|A|$ of the new octonion A was defined as $|A|^2 = A\overline{A}$. Similarly, we could have stated $|A|^2 = \overline{A}A$ because $A\overline{A} = \overline{A}A$.

Theorem 3

Unlike complex number, given two new octonions $\boldsymbol{A}$ and $\boldsymbol{B}$, we have

$$\boldsymbol{\overline{AB} \neq \overline{A}\,\overline{B},\ \overline{AB} = \overline{B}\,\overline{A}}.$$

The complex conjugates of two complex numbers $A = a + bi$ and $B = c + di$ are $\overline{A} = a - bi$ and $\overline{B} = c - di$, respectively. Thus, we have

$$\begin{aligned}\overline{A}\,\overline{B} &= (a - bi)(c - di)\\ &= ac - adi - bci + bdi^2\\ &= (ac - bd) - (ad + bc)i,\\ \overline{B}\,\overline{A} &= (c - di)(a - bi)\\ &= (ac - bd) - (ad + bc)i.\end{aligned}$$

Performing the calculations used in the proof of Theorem 1, we find that

$$AB = (ac - bd) + (ad + bc)i.$$

Thus, for complex numbers,

$$\overline{AB} = \overline{A}\,\overline{B} = \overline{B}\,\overline{A}.$$

By contrast, the complex conjugates of two new octonions $A = ah + bi + cj$ and $B = dh + ei + fj$ are $\overline{A} = ah - bi - cj$ and $\overline{B} = dh - ei - fj$, respectively. In this case, we have

$$\begin{aligned}\overline{A}\,\overline{B} &= (ah - bi - cj)(dh - ei - fj)\\ &= adh^2 - aehi - afhj - bdhi + bei^2 + bfij - cdhj + ceji + cfj^2\\ &= (-ad - be - cf) - (ae + bd)hi - (af + cd)hj + (bf - ce)k.\end{aligned}$$

Performing the calculations used in the proof of Theorem 1, we obtain

$$AB = (-ad - be - cf) + (ae + bd)hi + (af + cd)hj + (bf - ce)k.$$

Thus, we find that

$$\overline{AB} = (-ad - be - cf) - (ae + bd)hi - (af + cd)hj - (bf - ce)k.$$

Comparing $\overline{AB}$ with $\overline{A}\,\overline{B}$, the sign of k is reversed. Therefore, we have demonstrated that

$$\overline{AB} \neq \overline{A}\,\overline{B}.$$

Now, calculating $\overline{B}\,\overline{A}$, we obtain

$$\begin{aligned}\overline{B}\,\overline{A} &= (dh - ei - fj)(ah - bi - cj)\\ &= dah^2 - dbhi - dchj - eahi + ebi^2 + ecij - fahj + fbji + fcj^2\\ &= (-ad - be - cf) - (ae + bd)hi - (af + cd)hj - (bf - ce)k,\end{aligned}$$

which is identical to $\overline{AB}$, the product of two new octonions in the positive world. Thus, we have demonstrated that

$$\overline{AB} = \overline{B}\,\overline{A}.$$

The whole new octonion, containing the new octonion in the negative world, can be proved similarly, but the equations bocome more complex so the calculations are omitted.

Theorem 4
Given three new octonions $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, the associative law $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$ holds, as for complex numbers.

To prove this theorem, we calculate $(AB)C$ and $A(BC)$ assuming that $A = ah + bi + cj$, $B = dh + ei + fj$, and $C = lh + mi + nj$ (the calculations are lengthy and are hence omitted). Similarly, the associative law $(AB)C = A(BC)$ can be proved for the whole new octonion

$$ah + bi + cj + dk + p + qhi + rhj + shk$$

(containing the new octonion in the negative world), by rewriting it in the form

$$(ah + p) + (b + qh)i + (c + rh)j + (d + sh)k.$$

Theorem 5
Although the associative law $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$ is not realized in the Graves' octonion, it is realized in the new octonion.

This theorem paraphrases Theorem 4. Rewriting Theorem 4 in this way emphasizes the difference between the new octonion and the Graves' octonion. As is well-known, the Graves' octonion

$$a + b_1 i_1 + b_2 i_2 + b_3 i_3 + b_4 i_4 + b_5 i_5 + b_6 i_6 + b_7 i_7$$

does not satisfy the associative law $(AB)C = A(BC)$. Since new octonions do satisfy the associative law, they may be considered as natural numbers showing the properties of four-dimensional space–time.

Theorem 6

Like complex numbers, two new octonions $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy

$$|\boldsymbol{AB}|^{\mathbf{2}} = |\boldsymbol{A}|^{\mathbf{2}} |\boldsymbol{B}|^{\mathbf{2}} \,.$$

From $|A|^2 = A\overline{A}$ in Axiom 3, $\overline{AB} = \overline{B}\ \overline{A}$ in Theorem 3, and $(AB)C = A(BC)$ in Theorem 4, we obtain

$$\begin{aligned} |AB|^2 &= AB\overline{AB} \\ &= (AB)(\overline{B}\ \overline{A}) \\ &= A(B\overline{B}\ \overline{A}) \\ &= A\,|B|^2\,\overline{A} \\ &= A\overline{A}\,|B|^2 \\ &= |A|^2\,|B|^2 \,. \end{aligned}$$

We must not consider $\overline{AB} = \overline{A}\ \overline{B}$ from $|AB|^2 = |A|^2\,|B|^2$.

Theorem 7

The magnitude $|\boldsymbol{A}|$ of the new octonion $\boldsymbol{A}$ is either a positive real number or a positive imaginary number.

In calculations involving real and complex numbers, the magnitude is a positive real number or zero. However, the magnitude of a new octonion can also be a positive imaginary number. In complex numbers, if $A = i$, since $|A|^2 = A\overline{A}$ by Axiom 2, we have

$$\begin{aligned} |A| &= \sqrt{A\overline{A}} \\ &= \sqrt{i(-i)} \\ &= \sqrt{1} \\ &= 1. \end{aligned}$$

However, in the new octonion, if $A = h$, we have

$$\begin{aligned} |A| &= \sqrt{A\overline{A}} \\ &= \sqrt{hh} \\ &= h. \end{aligned}$$

That is, $|A|$ is a positive imaginary number. We should discard the prejudice that the magnitude must be a positive real number or zero, and instead regard the magnitude only as $|A|^2 = A\overline{A}$.

Theorem 8
Numeric solutions exist for the equation $|\boldsymbol{A}|^{\mathbf{2}} = \boldsymbol{\alpha}$.

This theorem implies that $A = \pm\sqrt{\alpha}$ does not follow from $|A|^2 = \alpha$. We consider the case of $|A|^2 = 4$. When A is a real number a satisfying $|A|^2 = 4$, we have

$$\begin{aligned} |A|^2 &= A\overline{A} \\ &= aa \\ &= a^2 \end{aligned}$$

from Axiom 2. Given that $|A|^2 = 4$, and a is a real number, we can write

$$\begin{aligned} a^2 &= 4, \\ a &= \pm\sqrt{4}, \\ a &= \pm 2. \end{aligned}$$

Thus, $A = \pm 2$.

Now let A be an imaginary number $a + bi$, where a and b are real. From Axiom 2, we have

$$\begin{aligned} |A|^2 &= A\overline{A} \\ &= (a + bi)(a - bi) \\ &= a^2 + b^2. \end{aligned}$$

Since $|A|^2 = 4$, we can write

$$a^2 + b^2 = 4.$$

Clearly, this formula will be satisfied by some combination of a and b. Therefore, when A is a complex number, there exists a numeric solution to $|A|^2 = \alpha$.

We next consider that A is a new octonion. For simplicity, we assume that $A = ah + bi$. From Axiom 2, we have

$$\begin{aligned} |A|^2 &= A\overline{A} \\ &= (ah + bi)(ah - bi) \\ &= -a^2 + b^2. \end{aligned}$$

Since $|A|^2 = 4$, we can write

$$-a^2 + b^2 = 4.$$

Again, this formula will be satisfied by some combination of the real numbers a and b. Thus, when A is a new octonion, there exists a numeric solution to $|A|^2 = \alpha$.

Theorem 9

In a given frame of reference, the relationships between two new octonions *A* and *B* can be investigated by comparing their coefficients. However, when viewed from different reference frames, we must instead compare the magnitudes of their coefficients.

Assume that the coordinates of a point mass D seen by a stationary observer A are $D(cth,\ xi)$. To observer B, moving in a straight line relative to the x-axial direction of A at uniform velocity v, the coordinates of D are $D(ct'h,\ x'i)$. As explained in Section 3.5, the new complex plane transforms as shown in Figure 13.1.

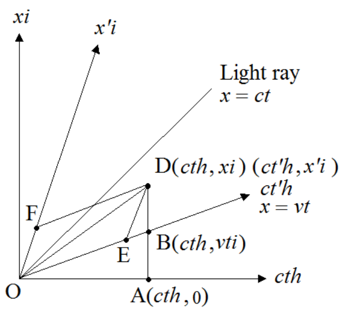


Figure 13.1

The equation $x = vt$ of the world line of B defines the $ct'h$-axis of B. The $x'i$-axis of B is the straight line symmetric about the world line of light, $x = ct$. The intercept of the $ct'h$-axis and straight line extending parallel to the $x'i$-axis from the point D, is designated point E. Similarly, the intercept of the $x'i$-axis and straight line extending parallel to the $ct'h$-axis from point D is designated point F.

The new octonion that describes D seen by observer A is $D_A = cth + xi$ while that describing D from observer B's viewpoint is $D_B = ct'h + x'i$. If $D_A = D_B$, we have

$$cth + xi = ct'h + x'i.$$

Comparing the coefficients, we obtain

$$ct = ct',\ \ x = x',$$

which are contradictory. The reason for this anomaly can be understood from Figure 13.1. From that figure, we have

$$cth = |OA|\,, x = |DA|\,,$$
$$ct'h = |OE|\,, x' = |OF|\,,$$

from which it is clear that, in general

$$|OA| \neq |OE|\,,\ |DA| \neq |OF|\,.$$

Since the imaginary numbers h of $D_A = cth + xi$ and $D_B = ct'h + x'i$ lie on separate coordinate axes, simply comparing their coefficients is inappropriate. The same conclusion can be drawn regarding the imaginary number i.

Although the coefficinets of D_A and D_B are not directly comparable, the relationship

$$|D_A| = |D_B|$$

holds, as clearly seen in Figure 13.1. Thus, this formula is appropriate for investigating the relationship between the coefficients of D_A and D_B.

When obtaining the new Lorentz transformations in Section 3.5, the coefficients of the imaginary numbers h and i were compared with no contradictions because the numbers resided on the the same coordinate axes.

Theorem 10

In two-dimensional space-time, rotation does not preserve similarity and congruence of figures

In Figure 13.2, the horizontal and vertical axes are designated the cth-axis and xi-axis, respectively. The respective coordinates of points A and B are $(h,\ 0)$ and $(h,\ i)$. The point B lies on the world line of light $x = ct$.

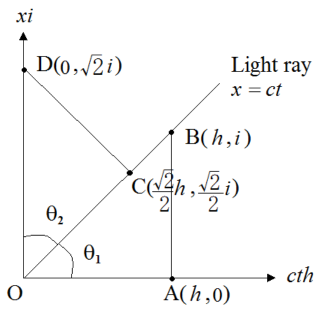


Figure 13.2

To realize Euclidean geometry in two-dimensional space–time, point A moves to point C along the line $x = ct$ when the triangle OAB is rotated 45° counterclockwise, while point B moves to D on the xi-axis. In Euclidean geometry, Pythagoras' theorem gives $|OB| = \sqrt{2}$. Thus, the coordinates of point D are $(0,\ \sqrt{2}i)$. In addition, since $|OC| = 1$, the coordinates of point C are $(\sqrt{2}h/2,\ \sqrt{2}i/2)$. Here, $|OB|$ and $|OC|$ express the distance between the origin and points B and C, respectively.

We next examine whether $\triangle OAB$ and $\triangle OCD$ are congruent, which seems to be the case in Euclidean geometry. The new octonions of each point are

$$A = h,\ B = h + i,\ C = \frac{\sqrt{2}}{2}h + \frac{\sqrt{2}}{2}i,\ D = \sqrt{2}i.$$

The squares of each world distance are

$$\begin{aligned}
|OA|^2 &= h^2 \\
&= -1, \\
|OB|^2 &= (h+i)(h-i) \\
&= h^2 - i^2 \\
&= -1 + 1 \\
&= 0, \\
|OC|^2 &= (\frac{\sqrt{2}}{2}h + \frac{\sqrt{2}}{2}i)(\frac{\sqrt{2}}{2}h - \frac{\sqrt{2}}{2}i) \\
&= \frac{1}{2}(h+i)(h-i) \\
&= \frac{1}{2}(h^2 - i^2) \\
&= \frac{1}{2}(-1+1) \\
&= 0, \\
|OD|^2 &= \sqrt{2}i(-\sqrt{2}i) \\
&= -2i^2 \\
&= 2.
\end{aligned}$$

Thus, since

$$\begin{aligned}
\frac{|OB|^2}{|OA|^2} &= \frac{0}{-1}, \\
\frac{|OD|^2}{|OC|^2} &= \frac{2}{0},
\end{aligned}$$

we have

$$\frac{|OB|^2}{|OA|^2} \neq \frac{|OD|^2}{|OC|^2}.$$

This result reveals that $\triangle OAB$ and $\triangle OCD$ are neither congruent nor similar. Thus, if a figure is rotated in two-dimensional space–time, the length of a side changes. In other words, the figures are not congruent despite their congruent appearance. This peculiar phenomenon is attributable to the curvature of two-dimensional space–time. In Einstein's general relativity, space is bent by mass. However, the above analysis suggests that four-dimensional space–time is inherently curved, even in the absence of mass. By the same method, the similarity of figures can be proven to be lost under rotation.

The space–time curve manifests from the relationship between time and space. We now demonstrate that congruence and similarity of figures are preserved after rotation in two-dimensional space. In Figure 13.3, the horizontal and vertical axes are designated the xi-axis and the yj-axis, respectively.

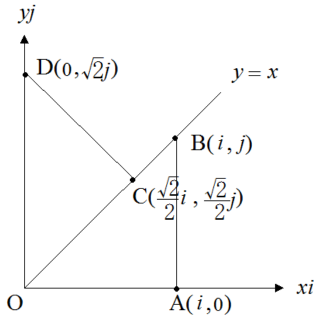


Figure 13.3

If the points A, B, C, and D are assigned as in Figure 13.2, their coordinates are

$$A(i,\ 0),\ B(i,\ j),\ C(\frac{\sqrt{2}}{2}i,\ \frac{\sqrt{2}}{2}j),\ D(0,\ \sqrt{2}j).$$

The new octonions are

$$A = i,\ B = i + j,\ C = \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j,\ D = \sqrt{2}j.$$

If the new octonions specifying sides AB and CD are written as (AB) and (CD), respectively, we have

$$\begin{aligned}(AB) &= B - A \\ &= i + j - i \\ &= j,\end{aligned}$$

$$
\begin{aligned}
(CD) &= D - C \\
&= \sqrt{2}j - \frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2}j \\
&= -\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j.
\end{aligned}
$$

Thus, because the squares of the world distances are

$$
\begin{aligned}
|OA|^2 &= i(-i) \\
&= 1, \\
|OB|^2 &= (i+j)(-i-j) \\
&= -i^2 - ij - ji - j^2 \\
&= 1 - k + k + 1 \\
&= 2, \\
|AB|^2 &= j(-j) \\
&= -j^2 \\
&= 1, \\
|OC|^2 &= (\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j)(-\frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2}j) \\
&= -\frac{1}{2}(i+j)(i+j) \\
&= -\frac{1}{2}(i^2 + ij + ji + j^2) \\
&= -\frac{1}{2}(i^2 + k - k + j^2) \\
&= 1, \\
|OD|^2 &= \sqrt{2}j(-\sqrt{2}j) \\
&= -2j^2 \\
&= 2, \\
|CD|^2 &= (-\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j)(\frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2}j) \\
&= -\frac{1}{2}(i-j)(i-j) \\
&= -\frac{1}{2}(i^2 - k + k + j^2) \\
&= 1,
\end{aligned}
$$

we can write

$$
|OA|^2 = |OC|^2\,,\ |OB|^2 = |OD|^2\,,\ |AB|^2 = |CD|^2\,.
$$

This result shows that $\triangle OAB$ and $\triangle OCD$ are congruent. In two-dimensional space, figures remain congruent after rotation in the *xi-yj, yj-zk*, and *zk-xi* planes. Similarity can be proven in the same manner. Congruency of figures is also realized under two-dimensional rotation in the negative world, i.e., under rotation in the *xhi-yhj, yhj-zhk*, and *zhk-xhi* planes.

The above results imply that the relationship between time and space is curved from the outset, and that space itself is flat. However, as will subsequently be proven in Theorem 15, if the positive and negative worlds are simultaneously considered, the space–space relationship becomes intrinsically curved.

Theorem 11
In two-dimensional space-time, congruency and similarity of figures are realized under parallel translation.

This theorem supplements Theorem 10. Since Theorem 11 is not a denial of a proposition, we prove it by a general approach.

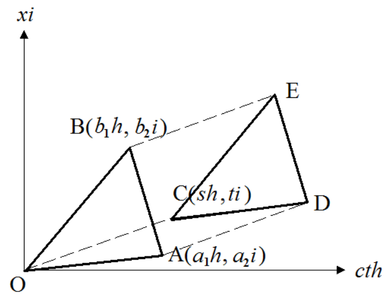


Figure 13.4

In Figure 13.4, the horizontal and vertical axes represent the cth-axis and xi-axis, respectively. The respective coordinates of points A and point B are $(a_1h,\ a_2i)$ and $(b_1h,\ b_2i)$. Now assume that $\triangle OAB$ is moved through a distance sh in the cth-direction and distance ti in the xi-direction (this movement constitutes a parallel translation). The translated figure is designated $\triangle CDE$. The new octonions of each point are

$$A = a_1h + a_2i,$$
$$B = b_1h + b_2i,$$
$$C = sh + ti,$$
$$D = a_1h + a_2i + sh + ti,$$

$$E = b_1h + b_2i + sh + ti.$$

In terms of the new octonions, sides OA, OB, CD, and CE are defined as (OA), (OB), (CD), and (CE), respectively. Explicitly, we have

$$\begin{aligned}
(OA) &= A = a_1h + a_2i,\\
(OB) &= B = b_1h + b_2i,\\
(CD) &= D - C\\
&= a_1h + a_2i + sh + ti - sh - ti\\
&= a_1h + a_2i,\\
(CE) &= E - C\\
&= b_1h + b_2i + sh + ti - sh - ti\\
&= b_1h + b_2i.
\end{aligned}$$

Thus, we can write

$$(OA) = (CD),\ (OB) = (CE).$$

Clearly from these relationships, $\triangle OAB$ and $\triangle CDE$ are congruent. Similarity can be proven by the same method.

Theorem 12

In two-dimensional space-time, if the gradients of the straight lines and curves that constitute a figure are invariant under translation, the translated figure is congruent and similar to the original figure. If the gradients change, congruency and similarity are violated.

This theorem summarizes Theorems 10 and 11.

Theorem 13

In two-dimensional space-time, arguments are not preserved under rotation.

In verifying Theorem 10, we found that

$$\frac{|OB|^2}{|OA|^2} \neq \frac{|OD|^2}{|OC|^2}.$$

Thus, we can state

$$\angle AOB \neq \angle COD$$

demonstrating that arguments are not preserved under rotation.

Therefore, we have avoided the word *angle* throughout this book (as mentioned in Section 1.2). Conformal mapping is a frequently used term in complex function theory. However, the curved nature of four-dimensional space–time renders it unsuitable for conformal mapping. Conformal mapping is entirely appropriate in complex number theory because the mathematics is constructed in two-dimensional non-curved spaces.

Theorem 14

Trigonometric function formulae are not realized in two-dimensional space-time.

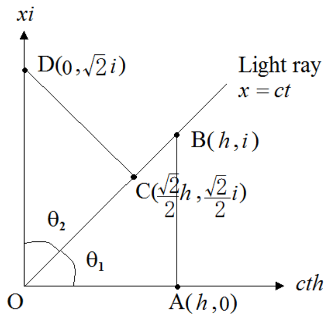


Figure 13.2

In Figure 13.2, used to prove Theorem 10, angles $\angle AOB$ and $\angle COD$ are denoted θ_1 and θ_2, respectively. In Euclidean space, $\theta_1 = \theta_2$. However, since

$$\cos\theta_1 = \frac{|OA|}{|OB|} = \frac{h}{0},$$
$$\cos\theta_2 = \frac{|OC|}{|OD|} = \frac{0}{\sqrt{2}},$$

the trigonometric functions are not realized in two-dimensional space–time. These functions are realized only in the three-dimensional space of the positive world. The adoption of triangular numbers in the mathematics and physics of four-dimensional space–time may lead to imprecise results. Similarly, a key equation in complex function theory, Euler's famous formula

$$e^{ix} = \cos x + i\sin x \tag{13.1}$$

comprises trigonometric functions, and is hence realized only in Euclidean flat space.

Replacing x with π in (13.1) yields one of the most ideal mathematical formulae

$$e^{i\pi} + 1 = 0. \tag{13.2}$$

Equation (13.2) is ideal because it links five fundamental mathematical constants

$$e,\ i,\ \pi,\ 0,\ 1$$

through a simple relationship. Although (13.2) is derived from trigonometric functions, it does not contain these functions, and is thus realizable in curved space–time. In Theorem 17, we shall prove that Pythagoras' theorem, which is applicable to flat space, is also realizable in curved two-dimensional space–time. It is expected that simple and ideal formulae will allow a general representation.

Theorem 15
Combining the positive and negative worlds admits two-dimensional space curves.

In verifying Theorem 10, we mentioned that when only the positive world is considered, distance is invariant under translation in two-dimensional space. However, if the positive and negative worlds are combined in two-dimesional space, this property is violated, as proven here.

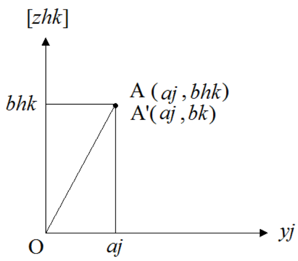


Figure 13.5

In Figure 13.5, we assume that the horizontal axis is the yj-axis in the positive world, while the vertical axis is the $[zhk]$-axis in the negative world. This coordinate plane was used in Section 12.3. If the coordinates of point A are $(aj,\ bhk)$, A becomes a point in the yj-$[zhk]$ plane described by the new octonion $A = aj + bhk$. The square of the length of A's world line is

$$|A|^2 = (aj + bhk)(-aj - bhk)$$

$$= -(aj)^2 - ajbhk - bhkaj - (bhk)^2$$
$$= a^2 - abhi + abhi - b^2$$
$$= a^2 - b^2. \tag{13.3}$$

Suppose that point A' in the positive world exists in the same location as point A with coordinates $(aj,\ bk)$. In this world, A' is described by the new octonion $A' = aj + bk$. The square of the world distance is

$$|A'| = (aj + bk)(-aj - bk)$$
$$= -(aj)^2 - abjk - bakj - (bk)^2$$
$$= a^2 - abi + abi + b^2$$
$$= a^2 + b^2. \tag{13.4}$$

In Equation (13.4), the negative world is disregarded. By contrast, (13.3) is the square of the world distance when the positive and negative worlds are considered together, or when each coordinate axis in four-dimensional space–time is regarded as a complex number. In other words, considering the whole four-dimensional space–time as a combination of positive and negative worlds, we achieve curvature in two-dimensional space.

As proven in Section 12.1, the *yj*-*zk* plane can be transformed into the *yj*-$[zhk]$ plane. Thus, while space does not curve in the stationary observer's world, it curves in the moving observer's world.

Theorem 16

If the case of realizing Pythagoras' theorem is defined as a rectangular cross, the rectangular condition is $m_1 m_2 = 1$, where m_1 and m_2 are the gradients of two straight lines in two-dimensional space-time. By contrast, in the Euclidean plane, the rectangular condition is $m_1 m_2 = -1$.

Pythagoras' theorem states that the squares of the shorter lengths, a and b, of a right-angled triangle sum to the square of the longest length d. Mathematically, Pythagoras' theorem is stated as $a^2 + b^2 = d^2$. As proven in Theorem 13, an argument is not preserved under rotation in two-dimensional space–time. Therefore, the phrase *right-angle* does not have a semantic. Consequently, if the intersection node of two straight lines and a point on each straight line can be connected to satisfy Pythagoras' theorem, we say that the two straight lines perpendicularly intersect. That is, rather than stating that a right-angled triangle satisfies Pythagoras' theorem, we consider that since Pythagoras' theorem is satisfied, two straight lines perpendicularly intersect. Because the argument lacks a semantic, we adopt the phrase *rectangular cross*.

First, we review the relationship between two straight lines perpendicularly intersecting on a flat two-dimensional surface. The straight lines perpendicularly intersecting through the origin are described by the equation $y = m_1 x$ and $y = m_2 x$ (see Figure 13.6).

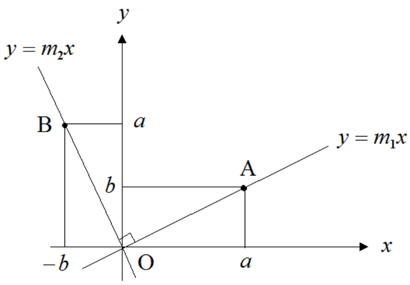


Figure 13.6

Since $\angle AOB$ is right-angled, if $m_1 = b/a,\ m_2 = -a/b$. Therefore, we obtain

$$
\begin{aligned}
m_1 m_2 &= \frac{b}{a} \times (-\frac{a}{b}) \\
&= -1. \qquad (13.5)
\end{aligned}
$$

(13.5) is the condition under which two straight lines perpendicularly intersect on a flat two-dimensional surface.

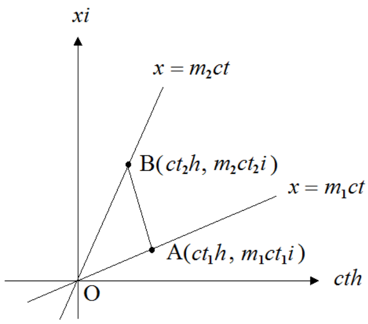


Figure 13.7

Next, we consider two-dimensional space–time. The situation is illustrated in Figure 13.7. Even under parallel translation, an argument is invariant from Theorem 11. Thus, to simplify the calculations, we assume the intersection node of two straight lines as the origin O. We denote the equation of a straight line by $x = m_1 ct$ and equation of the perpendicularly intersecting line by $x = m_2 ct$. It is important to note that the equation of the straight line of gradient m is not $x = mt$ but $x = mct$ because the unit of the horizontal axis is ct, not t.

We assume that the coordinates of point A on $x = m_1 ct$ are $(ct_1 h,\ m_1 ct_1 i)$, while those of point B on $x = m_2 ct$ are $(ct_2 h,\ m_2 ct_2 i)$. Points A and B are expressed by the new ocotnions $A = ct_1 h + m_1 ct_1 i$ and $B = ct_2 h + m_2 ct_2 i$. In addition, if the new octonion describing the side AB is written as (AB), we have

$$\begin{aligned}(AB) &= B - A \\ &= ct_2 h + m_2 ct_2 i - ct_1 h - m_1 ct_1 i \\ &= c(t_2 - t_1)h + c(m_2 t_2 - m_1 t_1)i.\end{aligned}$$

The squares of the world lengths are

$$\begin{aligned}|OA|^2 &= (ct_1 h + m_1 ct_1 i)(ct_1 h - m_1 ct_1 i) \\ &= c^2 t_1^2 h^2 - m_1^2 c^2 t_1^2 i^2 \\ &= -c^2 t_1^2 + m_1^2 c^2 t_1^2, \qquad (13.6)\\ |OB|^2 &= (ct_2 h + m_2 ct_2 i)(ct_2 h - m_2 ct_2 i) \\ &= c^2 t_2^2 h^2 - m_2^2 c^2 t_2^2 i^2 \\ &= -c^2 t_2^2 + m_2^2 c^2 t_2^2, \qquad (13.7)\\ |AB|^2 &= [c(t_2 - t_1)h + c(m_2 t_2 - m_1 t_1)i] \\ &\quad \times [c(t_2 - t_1)h - c(m_2 t_2 - m_1 t_1)i] \\ &= c^2 (t_2 - t_1)^2 h^2 - c^2 (m_2 t_2 - m_1 t_1)^2 i^2 \\ &= -c^2 (t_2 - t_1)^2 + c^2 (m_2 t_2 - m_1 t_1)^2. \qquad (13.8)\end{aligned}$$

If Pythagoras' theorem holds, we can write

$$|OA|^2 + |OB|^2 = |AB|^2 .$$

Substituting (13.6), (13.7), and (13.8) into this formula, we obtain

$$-c^2 t_1^2 + m_1^2 c^2 t_1^2 - c^2 t_2^2 + m_2^2 c^2 t_2^2 = -c^2 (t_2 - t_1)^2 + c^2 (m_2 t_2 - m_1 t_1)^2.$$

If this equation is solved further, we find

$$\begin{aligned}-c^2 t_1^2 + m_1^2 c^2 t_1^2 - c^2 t_2^2 + m_2^2 c^2 t_2^2 = &-c^2 t_2^2 + 2c^2 t_2 t_1 - c^2 t_1^2 \\ &+ c^2 m_2^2 t_2^2 - 2c^2 m_2 m_1 t_2 t_1 + c^2 m_1^2 t_1^2,\end{aligned}$$

which simplifies to

$$0 = 2c^2 t_2 t_1 - 2c^2 m_2 m_1 t_2 t_1.$$

If $t_2 \neq 0$ and $t_1 \neq 0$, both sides can be divided by $2c^2 t_2 t_1$ to yield

$$m_1 m_2 = 1.$$

Therefore, when two straight lines of gradients m_1 and m_2 perpendicularly intersect in two-dimensional space–time, they are related by

$$m_1 m_2 = 1. \tag{13.9}$$

This result reveals that Pythagoras' theorem is also realized in curved two-dimensional space–time. Equation (13.9) differs from Equation (13.5) only by a sign reversal.

Theorem 17

In two-dimensional space-time, if the gradients of two straight lines are line symmetric to the world line $x = ct$ or $x = -ct$, the intersection node and a point on each straight line can be connected to satisfy Pythagoras' theorem.

We now investigate the two straight lines that fulfill (13.9). As shown in Figure 13.8, if the coordinates of point A on the straight line $x = m_1 ct$ are $A(ah,\ bi)$, we have

$$m_1 = \frac{b}{a}.$$

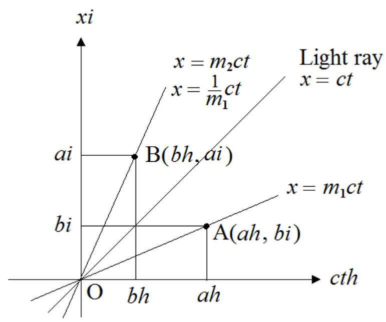


Figure 13.8

Substituting this relation into the rectangular condition

$$m_1 m_2 = 1 \tag{13.9}$$

in two-dimensional space–time, the equation becomes

$$\frac{b}{a}m_2 = 1,$$
$$m_2 = \frac{a}{b}.$$

This indicates that point $B(bh,\ ai)$ lies on the straight line $x = m_2ct$. From Figure 13.8, the gradient of the straight line $x = m_1ct$ with respect to the cth-axis is clearly that of the straight line $x = m_2ct$ with respect to the xi-axis. That is, the straight lines $x = m_1ct$ and $x = m_2ct$ are line-symmetric to the world line $x = ct$ and form oblique coordinate axes. If the two straight lines do not intersect at the origin O in two-dimensional space–time but if their gradients are line-symmetric to the world line $x = ct$, the intersection node and a point on each straight line can be connected to realize Pythagoras' theorem. This result shows the generality of Pythagoras' theorem. In addition, by the same method, it can be proven that two straight lines that are line-symmetric to the straight line $x = -ct$ perpendicularly intersect.

Theorem 18

Consider a two-dimensional plane, i.e., the yj-[zhk] or [yhj]-zk plane, constructed from space-coordinate axes of both positive and negative worlds. If the gradients of the two straight lines are m_1 and m_2, the rectangular condition is $m_1m_2 = 1$.

Since this theorem can be proven by the same method as Theorem 16, its verification is omitted. As shown in (13.5), $m_1m_2 = -1$ in the space frame of the positive world. However, in the yj-$[zhk]$ or $[yhj]$-zk plane, in which space coordinates in the positive and negative worlds are mixed like a mosaic, $m_1m_2 = 1$.

Theorem 19

An absolute rectangular frame does not exist.

From Theorem 17, Pythagoras' theorem is realized in both oblique and rectangular frames. Therefore, we cannot know whether our frame is right-angled or oblique. That is, oblique and right angles are relative.

In addition, because rectangular coordinates are considered as a static system, the fact that no absolute rectangular coordinates exist implies that no absolute rest frame exists. If points A and B are linearly moving at uniform velocities, their moving frames can be regarded as the rest frames.

Theorem 20

A straight line that is parallel to the straight line equidistant from both coordinate axes in the cth-xi plane in two-dimensional space-time has a world distance of zero.

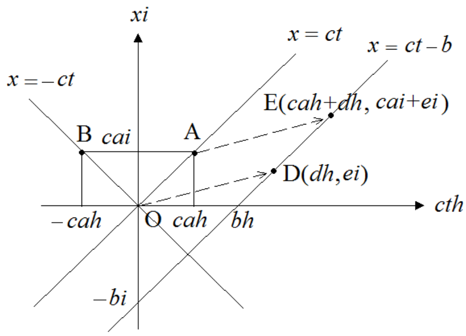


Figure 13.9

The equation of the world line of light, which is equidistant from the cth-axis and xi-axis (see Figure 13.9) is $x = ct$. The coordinates of point A on this line are $(cah,\ cai)$, and the new octonion describing A is $A = cah + cai$. Thus, we have

$$\begin{aligned}|A|^2 &= (cah + cai)(cah - cai)\\ &= (cah)^2 - (cai)^2\\ &= -c^2a^2 + c^2a^2\\ &= 0,\end{aligned}$$

as explained in Section 8.2.

Next, the origin O and point A are moved to points D and E, respectively, which lie on the straight line $x = ct - b$ that is parallel to $x = ct$ (i.e., has the same gradient). If the coordinates of point D are $(dh,\ ei)$, the coordinates of point E are $(cah + dh,\ cai + ei)$. The new octonions of points D and E are

$$D = dh + ei,\ \ E = (ca + d)h + (ca + e)i.$$

If the new octonion describing the line segment DE is written (DE), we have

$$\begin{aligned}(DE) &= E - D\\ &= (ca + d)h + (ca + e)i - (dh + ei)\\ &= cah + cai.\end{aligned}$$

Since the same octonion describes A, we can write

$$|DE|^2 = |A|^2$$
$$= 0.$$

Therefore, in the *cth*-*xi* plane in two-dimensional space–time, the world distance of a straight line parellel to the straight line equidistant from both positive coordinate axes is zero.

We now calculate the world distance of point B on the straight line $x = -ct$, which is equidistant from both the $-cth$-axis and xi-axis in Figure 13.9. Since the new octonion of B is $B = -cah + cai$, we obtain

$$\begin{aligned} |B|^2 &= (-cah + cai)(-cah - cai) \\ &= (cah)^2 - (cai)^2 \\ &= -c^2a^2 + c^2a^2 \\ &= 0. \end{aligned}$$

Therefore, Theorem 20 is also realized in the negative region of the coordinate axis.

Theorem 21
In the two-dimensional mosaic space, comprising space-coordinate axes in the positive and negative worlds, such as the yj-[zhk] and [yhj]-zk planes, the straight line parallel to the line equidistant from both coordinate axes has a world distance of zero.

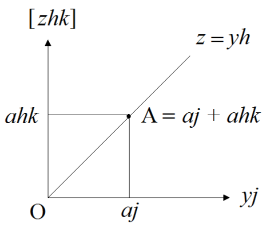


Figure 13.10

To prove this theorem, we consider point A on the straight line $z = yh$ that is equidistant from both coordinate axes in the yj-$[zhk]$ plane. The situation is illustrated in Figure 13.10. Since the new octonion is $A = aj + ahk$, we obtain

$$\begin{aligned} |A|^2 &= (aj + ahk)(-aj - ahk) \\ &= -(aj + ahk)(aj + ahk) \end{aligned}$$

$$= -a^2j^2 - 2a^2hjk - 2a^2hkj - a^2h^2k^2$$
$$= a^2 - 2a^2hi + 2a^2hi - a^2$$
$$= 0.$$

Therefore, similarly to Theorem 20, it is proven that the world distance of the straight line parallel to the line equidistant from both coordinate axes is zero.

Theorem 22

Multiplying by i is identical to a counterclockwise 90° rotation in two-dimensional space-time.

As explained in Section 1.2, multiplying by i in the complex plane performs a counterclockwise 90° rotation. We now prove that this operation genertes the same result in the new complex plane, the *cth-xi* plane. In this proof, we must assume the double structure of four-dimensional space–time.

We consider the two-dimensional space–time, in which the positive and negative worlds overlap, as shown in Figure 13.11. If the coordinates of point A in the positive world are $(ah,\ bi)$, the new octonion describing A is $A = ah + bi$.

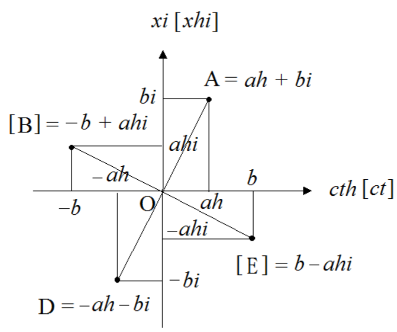


Figure 13.11

We systematically multiply each term in the new octonion by i. The symbol [ ] denotes a point in the negative world. We find that

$$Ai = (ah + bi)i = ahi + bi^2 = -b + ahi = [B],$$
$$Ai^2 = [B]i = (-b + ahi)i = -ah - bi = D = -A,$$
$$Ai^3 = [B]i^2 = Di = (-ah - bi)i = b - ahi = [E] = -[B],$$
$$Ai^4 = [B]i^3 = Di^2 = [E]i = (b - ahi)i = ah + bi = A.$$

According to this result, multiplication by i is equivalent to a 90° counterclockwise rotation of a point. In addition, the point cyclically enters and exits the positive and negative worlds. This theorem is realized only if the double structure of four-dimensional space–time is assumed.

Above, we proved the theorem in two-dimensional space–time. We now examine the result of multiplying by i in four-dimensional space–time.

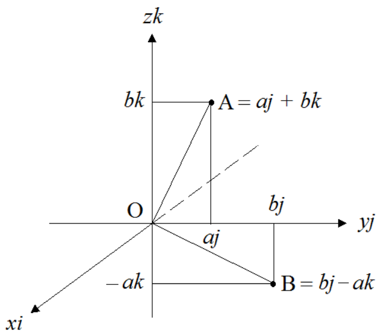


Figure 13.12

We consider point $A = aj + bk$ on the yj-zk plane, as shown in Figure 13.12. Multiplying A by i, we obtain

$$\begin{aligned} Ai &= (aj + bk)i \\ &= aji + bki \\ &= -ak + bj \\ &= bj - ak. \end{aligned}$$

Denote the resultant point by B. B is the result of rotating point A by 90° in the reverse direction of a direction to that of a right-hand screw advancing in the positive direction of the xi-axis. Although not illustrated, if A lies on the xi-yj plane, i.e., $A = ai + bj$, multiplication by i gives

$$\begin{aligned} Ai &= (ai + bj)i \\ &= -a - bk. \end{aligned}$$

The time component $-a$ of this resultant lies in the negative world, while the z-axis component $-b$ lies in the positive world.

Theorem 23
Multiplying by h is equivalent to a line-symmetric translation along the coordinate axes in two-dimensional space-time.

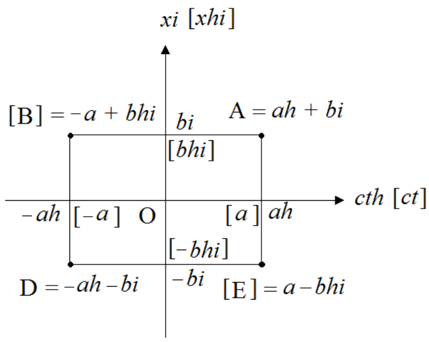


Figure 13.13

To prove this theorem, we require the two-dimensional space–time in which the positive and negative worlds overlap. This situation is illustrated in Figure 13.13. If the coordinates of point A in the positive world are given by $(ah,\ bi)$, the new octonion is $A = ah + bi$. We systematically multiply each term in the new octonion by h.

$$\begin{aligned}
Ah &= (ah + bi)h = ah^2 + bhi = -a + bhi = [B],\\
Ah^2 &= [B]h = (-a + bhi)h = -ah - bi = D = -A,\\
Ah^3 &= [B]h^2 = Dh = (-ah - bi)h = a - bhi = [E] = -[B],\\
Ah^4 &= [B]h^3 = Dh^2 = [E]h = (a - bhi)h = ah + bi = A.
\end{aligned}$$

Thus, multiplication by h performs a line-symmetric translation along the coordinate axes. The point cyclically enters and exits the positive and negative worlds. Like Theorem 22, Theorem 23 is realized only when the double structure of four-dimensional space–time is assumed.

Theorem 24

Multiplying by $i/h = -hi$ is equivalent to a line-symmetric translation along the world line of light in two-dimensional space-time.

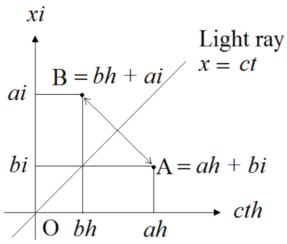


Figure 13.14

Here, we consider two-dimensional space–time in the positive world, as shown in Figure 13.14. If the coordinates of point A in the positive world are $(ah,\ bi)$, the new octonion is $A = ah + bi$. We systematically multiply each term in the new octonion by i/h. We find that

$$A(i/h) = A(-hi) = (ah + bi)(-hi) = -bhi^2 - ah^2 i = bh + ai = B,$$
$$A(i/h)^2 = A(-hi)^2 = B(-hi) = (bh + ai)(-hi) = ah + bi = A.$$

Therefore, multiplication by $i/h = -hi$ performs a line-symmetric translation alomg the world line of light. In this case, the point moves only in the positive world. Similarly, multiplying point $A = a + bhi$ in the negative world by $i/h = -hi$ is equivalent to a line-symmetrc translation along the world line of light in the negative world.

By the same method, we can prove that multiplication by hi performs a line-symmetric translation along the world line $x = -ct$.

Theorem 25

Both sides of an equality cannot be divided by a new octonion whose absolute value is zero.

Consider the following equality:

$$(h + i)hi = h^2 i + hi^2 = -(h + i). \tag{13.10}$$

Assuming that both sides of (13.10) are divisible by $(h + i)$, we find that

$$hi = -1.$$

This result contradicts Axiom 1, which implies that hi is a fundamental number and cannot be replaced by other numbers. Therefore, both sides of the equality are not divisible by $(h + i)$.

We now discuss the reason for this result. In Axiom 4, the division $A \div B$ of two new octonions A and B is given by

$$A \div B = \frac{A\overline{B}}{|B|^2}.$$

That is, the new octonion cannot be divided by $(h+i)$, but instead must be divided by $|h + i|^2$. Then, the equation becomes

$$\begin{aligned} |h + i|^2 &= (h + i)(h - i) \\ &= h^2 - i^2 \\ &= 0. \end{aligned}$$

Since mathematics forbids the division of both sides of an equality by zero, both sides of (13.10) cannot be divided by $(h+i)$. Besides $(h+i)$, there exist many new octonions, for example $(h+j)$, $(h+k)$, and $(h-i)$, whose absolute values are zero.

Theorem 26

The derivative world distance *dl* in two-dimensional space-time is

$$\boldsymbol{dl = chdt\sqrt{1-(dx)^2/(cdt)^2}.}$$

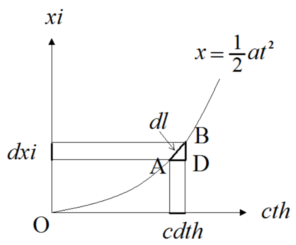


Figure 13.15

As shown in Figure 13.15, the equation of the world line of a point mass accelerating in the x-direction of a rest frame in two-dimensional space–time is $x = at^2/2$, where a is the mass's acceleration. However, in relativity theory, a is treated as a constant only at slow velocities. The case of varying a is treated in Section 20.2. In addition, though $(dl)^2$ is usually written as dl^2 in calculus, we adopt the $(dl)^2$ notation for the benefit of readers who lack a calculus background.

Suppose that two points A and B approach extremely close on the world line $x = at^2/2$. The infinitesimal time and distance between A and B are dt and dx, respectively. Since the new octonion describing the infinitesimal line segment AB is $cdth + dxi$, the square of the infinitesimal world distance dl is

$$\begin{aligned}(dl)^2 &= (cdth + dxi)(cdth - dxi)\\ &= c^2(dt)^2h^2 - (dx)^2i^2.\end{aligned} \tag{13.11}$$

When $cdt > dx$, i.e., $c > dx/dt = v$, the square root is rendered positive by extracting the term $cdth$, as explained in Axiom 9. Thus, we have

$$\begin{aligned}dl &= \sqrt{c^2(dt)^2h^2 - (dx)^2i^2}\\ &= cdth\sqrt{1-(dxi)^2/(cdth)^2}\\ &= chdt\sqrt{1-(dx)^2/(cdt)^2}.\end{aligned} \tag{13.12}$$

This formula gives the derivative world distance in two-dimensional space–time. It is important to note that, unlike flat space–time, $(dl)^2$ cannot be

$$c^2(dt)^2h^2 + (dx)^2i^2$$

or

$$c^2(dt)^2 + (dx)^2$$

in (13.11). From the equation $x = at^2/2$ of the world line, we find that

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}(at^2/2)\\ &= at.\end{aligned}$$

Substituting this expression into (13.12), we obtain

$$dl = chdt\sqrt{1-(at/c)^2}.$$

When we assume that the coordinates of A are $(ct_0,\ x_0)$, the distance l of the world line from the origin O to point A is

$$\begin{aligned}l &= \int_0^{t_0} chdt\sqrt{1-(at/c)^2}\\ &= ch\int_0^{t_0}\sqrt{1-(at/c)^2}dt.\end{aligned} \tag{13.13}$$

Equation (13.13) gives the world distance of curvilinear $x = at^2/2$ in two-dimensional space–time.

We now determine the infinitesimal distance in a timeless three-dimensional space. If two points A and B approach extremely closely in this three-dimensional space, the infinitesimal components in the x-, y-, and z-directions are dx, dy, and dz, respectively. Since the new octonion describing the line segment AB is $dxi + dyj + dzk$, the square of the infinitesimal distance dl is

$$\begin{aligned}(dl)^2 &= (dxi + dyj + dzk)(-dxi - dyj - dzk)\\ &= -(dxi)^2 - (dyj)^2 - (dzk)^2\\ &= (dx)^2 + (dy)^2 + (dz)^2,\\ dl &= \sqrt{(dx)^2 + (dy)^2 + (dz)^2}.\end{aligned} \tag{13.14}$$

Equation (13.14) is exactly the infinitesimal distance between two points in flat three-dimensional space, derived from calculus.

Theorem 27

If the new octonions of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are specified as A and B, respectively, their inner product $\boldsymbol{A}\cdot\boldsymbol{B}$ and outer product $\boldsymbol{A}\times\boldsymbol{B}$ are given by

$$B\overline{A} = \boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A}\times\boldsymbol{B}.$$

This theorem will be proved in Section 14.2.

Theorem 28

If the new octonions of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are specified by A and B, respectively, their inner product $\boldsymbol{A}\cdot\boldsymbol{B}$ and outer product $\boldsymbol{A}\times\boldsymbol{B}$ are given by

$$\boldsymbol{A}\cdot\boldsymbol{B} = (B\overline{A} + A\overline{B})/2,$$
$$\boldsymbol{A}\times\boldsymbol{B} = (B\overline{A} - A\overline{B})/2.$$

This theorem will be proved in Section 14.4.

Theorem 29

If the new octonions of three vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are specified by A, B, and C, respectively, their triple scalar product $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$ and triple vector product $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})$ are given by

$$(C\overline{B} - B\overline{C})\overline{A}/2 = \boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) + \boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}).$$

This theorem will be proved in Section 14.4.

Theorem 30

New octonion geometry obeys non-Euclidean and non-Riemannian geometry.

This theorem does not appear in the revised Japanese second edition.

There is no bend in space and parallel lines do not cross in Euclidean geometry. Space curves and parallel lines cross in Riemannian geometry. However, although space curves, parallel lines do not cross in the new octonion geometry, i.e., the new octonion geometry is the third geometry. We prove this here.

Section 13.1 explained that in the process of proving the fifth axiom (postulate) of Euclid, i.e., parallel lines do not cross, we discovered Riemannian geometry where parallel lines do cross. In addition, it was found that space curves in the Riemann space where parallel lines cross. Einstein's general relativity is based on Riemannian geometry. Four-dimensional space–time is bent in the new octonion geometry as proven in Section 8.2 and by Theorem 15. Do parallel lines cross in this curved space–time? If parallel lines cross, the new octonion geometry is a Riemannian geometry. If they do not cross, it is a new geometry.

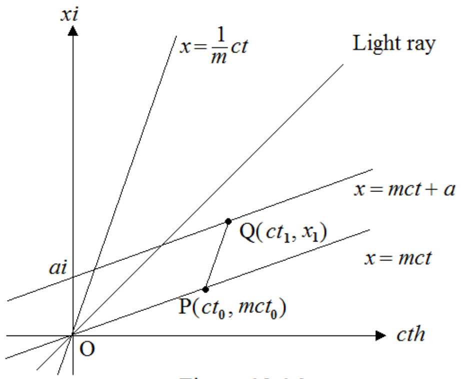


Figure 13.16

As shown in Figure 13.16, we assume that the straight lines $x = mct$ and $x = mct + a$ lie in the *cth*-*xi* new complex plane. As explained in Section 5.2, because a straight line in the new complex plane can be calculated with the equation of a real number, the following calculations are done with real numbers. Point $P(ct_0,\ mct_0)$ is on the straight line $x = mct$ and point $Q(ct_1,\ mct_1 + a)$ is on the straight line $x = mct + a$. It was proved in Theorem 17 that when the Pythagorean theorem is applied, two straight lines are line symmetric about the world line $x = ct$ of light. Therefore, if the straight line PQ and the straight line $x = mct$ intersect rectangularly, the gradient of another oblique axis of $x = mct$ and the gradient of the straight line PQ are the same. As explained in Section 4.3, the equation of another oblique axis of $x = mct$ can be obtained by replacing x and ct by ct and x, respectively. If we actually calculate, we have

$$ct = mx,$$
$$x = \frac{1}{m}(ct).$$

Since the units of the *cth*-axis are *ct*, the gradient of the oblique axis is $1/m$. Therefore, from the condition that the straight line passes point P, the equation of the straight line PQ is

$$x - mct_0 = \frac{1}{m}(ct - ct_0).$$

Because this straight line passes point Q, we can write

$$(mct_1 + a) - mct_0 = \frac{1}{m}(ct_1 - ct_0).$$

By solving this equation, we find that

$$\begin{aligned}(m - \frac{1}{m})ct_1 &= (m - \frac{1}{m})ct_0 - a,\\ ct_1 &= ct_0 - a/(m - \frac{1}{m}),\\ t_1 &= t_0 - \frac{ma}{(m^2-1)c}.\end{aligned} \tag{13.15}$$

We assume that the x-axial component of point Q is x_1. By substituting (13.15) into $x = mct + a$, we have

$$\begin{aligned}x_1 &= mc[t_0 - ma/(m^2c - c)] + a\\ &= mct_0 - \frac{m^2a}{m^2-1} + a\\ &= mct_0 + \frac{-m^2a + m^2a - a}{m^2-1}\\ &= mct_0 - \frac{a}{m^2-1}.\end{aligned}$$

If we assume that the new octonions showing the line segments OP, OQ, and PQ are (OP), (OQ), and (PQ), respectively, we find that

$$\begin{aligned}(OP) &= ct_0h + mct_0i,\\ (OQ) &= ct_1h + x_1i\\ &= [ct_0 - ma/(m^2-1)]h + [mct_0 - a/(m^2-1)]i,\\ (PQ) &= (OQ) - (OP)\\ &= [ct_0 - ma/(m^2-1) - ct_0]h + [mct_0 - a/(m^2-1) - mct_0]i\\ &= -mah/(m^2-1) - ai/(m^2-1).\end{aligned}$$

Thus, we can write

$$\begin{aligned}|PQ|^2 &= [mah/(m^2-1) + ai/(m^2-1)][mah/(m^2-1) - ai/(m^2-1)]\\ &= [ma/(m^2-1)]^2h^2 - [a/(m^2-1)]^2i^2\end{aligned}$$

$$= \frac{-m^2a^2}{(m^2-1)^2} + \frac{a^2}{(m^2-1)^2}$$
$$= \frac{-(m^2-1)a^2}{(m^2-1)^2}$$
$$= \frac{-a^2}{m^2-1}. \tag{13.16}$$

Because m and a are constant, from (13.16), it is proved that regardless of the location of point $P(ct_0,\ mct_0)$, $|PQ|^2$ is constant. From the above result, because the new octonion space–time is bent but parallel lines do not cross there, the new octonion geometry is a non-Euclidean geometry and non-Riemannian geometry, respectively.

Table 13.1

bend of space-time \ cross of parallel lines	(–)	(+)
(–)	Euclid	(#)
(+)	new octonion	Riemann

As shown in Table 13.1, four kinds of geometry can be considered from the intersection of parallel lines and the bend of space–time. Of these, the geometry of the region (#) will not be found because Riemannian geometry with curved space was found in the process of seeking the geometry where parallel lines cross in a flat plane, as explained in Section 13.1. If our universe has the structure indicated by the new octonion geometry, the present cosmology obtained using Riemannian geometry needs to be reconsidered.

14

New Octonion and Vectors

14.1 Basic properties of vectors

In this chapter, we prove that vector calculations can be reformulated as new octonions. To prepare the reader, we first review the basic properties of vectors. Quantities with magnitude and no direction, such as length, time, and mass, are called scalars. Scalars are expressed only as numbers. On the other hand, quantities with both magnitude and direction, such as velocity, acceleration and force, are called vectors. Vectors are represented by arrows whose length and orientation indicate their magnitude and direction, respectively. Vectors are expressed in bold font; for example, $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\cdots$, and their magnitudes are expressed as absolute values $|\boldsymbol{A}|$, $|\boldsymbol{B}|$, $\cdots$.

Vector can be multiplied using two methods; as an inner product and as an outer product. Given two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ on a plane, the inner product is denoted by $\boldsymbol{A}{\cdot}\boldsymbol{B}$ and the outer product is denoted by $\boldsymbol{A}\times\boldsymbol{B}$. These multiplications are visually interpreted in Figure 14.1.

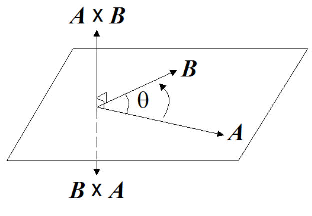


Figure 14.1

An inner product is a scalar. If we assume that the angle between two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is θ (theta), the inner product is defined by

$$\boldsymbol{A}{\cdot}\boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}|\cos\theta. \tag{14.1}$$

Since this result is a scalar, it has no direction. The outer product is a vector of magnitude $|\boldsymbol{A} \times \boldsymbol{B}|$, defined by

$$|\boldsymbol{A} \times \boldsymbol{B}| = |\boldsymbol{A}|\,|\boldsymbol{B}| \sin\theta. \tag{14.2}$$

The direction of the outer product is perpendicular to the plane described by the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. In Figure 14.1, where $\boldsymbol{A}$ and $\boldsymbol{B}$ lie in the horizontal plane, the outer product is oriented both upward and downward. By convention, the positive orientation of the vector $\boldsymbol{A} \times \boldsymbol{B}$ is defined as the direction of a right-handed screw rotating from vector $\boldsymbol{A}$ to vector $\boldsymbol{B}$. However, this convention is quite arbitrary; we could also define the opposite orientation as positive. Regarding $\boldsymbol{B} \times \boldsymbol{A}$, when a right-handed screw rotates from $\boldsymbol{B}$ to $\boldsymbol{A}$, it follows a direction opposite to $\boldsymbol{A} \times \boldsymbol{B}$. Thus, we can state

$$\boldsymbol{B} \times \boldsymbol{A} = -\boldsymbol{A} \times \boldsymbol{B}.$$

Why the outer product is oriented perpendicular to the plane described by the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is not explained in mathematics and physics texts. The definition is provided while the rationale is not explained, probably because the calculations show no contradictions; therefore, the definition is assumed true without mathematical proof. For this reason, when the vector was proposed at the end of the 19th century, it was regarded by some mathematicians as a non-mathematical construct. The direction of the outer product will be discussed in Section 14.3.

In addition, since the positive and negative directions of $\boldsymbol{A} \times \boldsymbol{B}$ may be arbitrarily chosen, the right-hand screw rule that determines the positive direction of $\boldsymbol{A} \times \boldsymbol{B}$ has no mathematical basis. If the positive direction were redefined as the direction followed by a left-handed screw, the mathematics would remain consistent. Therefore, vector operations are not admitted as mathematics by those who insist on strict theorems based on minimum axioms. Euclid, who wrote *Elements*, will not acknowledge the arbitrariness of the screw rule that a right-handed screw or a left-handed screw can be used.

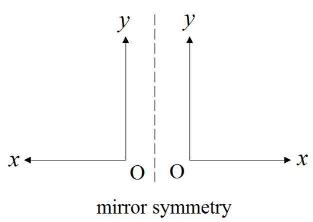


Figure 14.2

Since this concept is not discussed in conventional texts, we here provide a mathematical basis for the freedom of the positive and negative directions. We consider the x-y coordinates of a two-dimensional plane (see Figure 14.2). The coordinate axes can be decided in two ways. If upward is selected as the positive direction of the y-axis, the positive direction of the x-axis can be considered as either leftward or rightward. The two coordinate systems are mirror symmetric.

Hypothetical creatures inhabiting a two-dimensional world cannot add height to these two coordinate planes. Because three-dimensional space does not exist in their world, they can shift a geometric figure only within the coordinate plane. In contrast, life forms inhabiting three-dimensional space can fold two coordinate planes into a three-dimensional structure. That is, while our hypothetical two-dimensional counterparts view the x-y coordinate planes as different, we regard them as the same.

This view is extendable to three-dimensional space and four-dimensional space–time. In three-dimensional space, the direction of $\boldsymbol{A} \times \boldsymbol{B}$ may be either up or down. Since they are mirror symmetrc, we view them as different. However, since both directions of $\boldsymbol{A} \times \boldsymbol{B}$ can be piled up in four-dimensional space–time, they are actually the same.

In Section 11.1, it was explained that only three kinds of space–times; one-dimensional, two-dimensional, and four-dimensional, are mathematically possible. Since three-dimensional space cannot independently exist, we must consider the vector in four-dimensional space–time. Relativity theory asides, contemporary mathematics and physics treat the vector in three-dimensional space. Thus, two apparent directions for $\boldsymbol{A} \times \boldsymbol{B}$ exist. However, from the viewpoint of four-dimensional space–time, a single direction exists. That is, since $\boldsymbol{A} \times \boldsymbol{B}$ exists in four-dimensional space–time, the mathematics is consistent if the positive and negative directions of an outer product are reveresed in three-dimensional space.

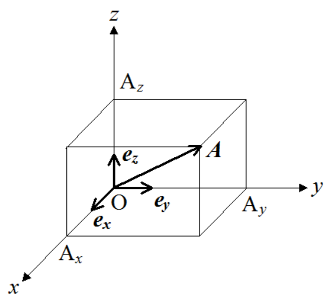


Figure 14.3

Next, we explain how a vector is described by its coordinate components. The vectors of unit length 1 along the x-, y-, z-axes in three-dimensional space are called fundamental vectors, expressed as $\boldsymbol{e_x}$, $\boldsymbol{e_y}$, and $\boldsymbol{e_z}$. Then, the x-, y-, and z-axial components of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are denoted A_x, A_y, A_z and B_x, B_y, B_z, respectively. In terms of the unit vectors, these are expressed as

$$\boldsymbol{A} = A_x\boldsymbol{e_x} + A_y\boldsymbol{e_y} + A_z\boldsymbol{e_z},\ \boldsymbol{B} = B_x\boldsymbol{e_x} + B_y\boldsymbol{e_y} + B_z\boldsymbol{e_z}. \tag{14.3}$$

The unit vector representation of A is illustrated in Figure 14.3. From the definition of the inner product, i.e.,

$$\boldsymbol{A}\cdot\boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}|\cos\theta, \tag{14.1}$$

and given that $\cos 0^\circ = 1$, $\cos 90^\circ = 0$, the inner products of the fundamental vectors $\boldsymbol{e_x}$, $\boldsymbol{e_y}$, and $\boldsymbol{e_z}$ are

$$\boldsymbol{e_x}\cdot\boldsymbol{e_x} = 1,\ \boldsymbol{e_y}\cdot\boldsymbol{e_y} = 1,\ \boldsymbol{e_z}\cdot\boldsymbol{e_z} = 1, \tag{14.4}$$

$$\boldsymbol{e_x}\cdot\boldsymbol{e_y} = 0,\ \boldsymbol{e_y}\cdot\boldsymbol{e_z} = 0,\ \boldsymbol{e_z}\cdot\boldsymbol{e_x} = 0. \tag{14.5}$$

Therefore, from (14.3), (14.4), and (14.5), the inner product $\boldsymbol{A}\cdot\boldsymbol{B}$ is expressed in terms of its unit vectors as

$$\begin{aligned}\boldsymbol{A}\cdot\boldsymbol{B} &= (A_x\boldsymbol{e_x} + A_y\boldsymbol{e_y} + A_z\boldsymbol{e_z})\cdot(B_x\boldsymbol{e_x} + B_y\boldsymbol{e_y} + B_z\boldsymbol{e_z})\\ &= A_xB_x + A_yB_y + A_zB_z.\end{aligned} \tag{14.6}$$

Similarly, from the definition of the outer product, i.e.,

$$|\boldsymbol{A}\times\boldsymbol{B}| = |\boldsymbol{A}|\,|\boldsymbol{B}|\sin\theta, \tag{14.2}$$

and given that $\sin 0^\circ = 0$, $\sin 90^\circ = 1$, and the right-hand screw rule gives the outer products of the fundamental vectors $\boldsymbol{e_x}$, $\boldsymbol{e_y}$, and $\boldsymbol{e_z}$ as

$$\boldsymbol{e_x}\times\boldsymbol{e_x} = 0,\ \boldsymbol{e_y}\times\boldsymbol{e_y} = 0,\ \boldsymbol{e_z}\times\boldsymbol{e_z} = 0, \tag{14.7}$$

$$\boldsymbol{e_x}\times\boldsymbol{e_y} = \boldsymbol{e_z},\ \boldsymbol{e_y}\times\boldsymbol{e_z} = \boldsymbol{e_x},\ \boldsymbol{e_z}\times\boldsymbol{e_x} = \boldsymbol{e_y}, \tag{14.8}$$

$$\boldsymbol{e_y}\times\boldsymbol{e_x} = -\boldsymbol{e_z},\ \boldsymbol{e_z}\times\boldsymbol{e_y} = -\boldsymbol{e_x},\ \boldsymbol{e_x}\times\boldsymbol{e_z} = -\boldsymbol{e_y}. \tag{14.9}$$

Therefore, from (14.3), (14.7), (14.8), and (14.9), the outer product $\boldsymbol{A}\times\boldsymbol{B}$ is expressed in terms of unit vectors as

$$\begin{aligned}\boldsymbol{A}\times\boldsymbol{B} &= (A_x\boldsymbol{e_x} + A_y\boldsymbol{e_y} + A_z\boldsymbol{e_z})\times(B_x\boldsymbol{e_x} + B_y\boldsymbol{e_y} + B_z\boldsymbol{e_z})\\ &= A_xB_y\boldsymbol{e_x}\times\boldsymbol{e_y} + A_xB_z\boldsymbol{e_x}\times\boldsymbol{e_z}\\ &\quad + A_yB_x\boldsymbol{e_y}\times\boldsymbol{e_x} + A_yB_z\boldsymbol{e_y}\times\boldsymbol{e_z}\end{aligned}$$

$$
\begin{aligned}
&\quad + A_z B_x \boldsymbol{e_z} \times \boldsymbol{e_x} + A_z B_y \boldsymbol{e_z} \times \boldsymbol{e_y} \\
&= A_x B_y \boldsymbol{e_z} - A_x B_z \boldsymbol{e_y} \\
&\quad - A_y B_x \boldsymbol{e_z} + A_y B_z \boldsymbol{e_x} \\
&\quad + A_z B_x \boldsymbol{e_y} - A_z B_y \boldsymbol{e_x} \\
&= (A_y B_z - A_z B_y)\boldsymbol{e_x} + (A_z B_x - A_x B_z)\boldsymbol{e_y} + (A_x B_y - A_y B_x)\boldsymbol{e_z}.
\end{aligned}
$$

Summarizing this result, we obtain

$$
\boldsymbol{A} \times \boldsymbol{B} = (A_y B_z - A_z B_y)\boldsymbol{e_x} + (A_z B_x - A_x B_z)\boldsymbol{e_y} + (A_x B_y - A_y B_x)\boldsymbol{e_z}. \quad (14.10)
$$

As explained previously, the vector must be considered in four-dimensional space–time. However, since the angles made by the fundamental vector $\boldsymbol{e_t}$ on the temporal axis and the fundamental vectors $\boldsymbol{e_x}$, $\boldsymbol{e_y}$, and $\boldsymbol{e_z}$ in three-dimensional space are unknown, the formulae describing the inner and outer products in four-dimensional space–time by their coordinate components cannot be verified as correct. In Section 14.6, we show that the inner and outer products in four-dimensional space–time can be expressed in coordinate components using the new octonion.

14.2 Vectors and coordinate transformations

In this section, we prove that vector multiplications (inner and outer products) are coordinate transformations. Multiplying vector A by vector B is equivalent to changing vector B into the coordinate system of vector A. In addition, we prove that since vector multiplication is a coordinate transformation, both inner and outer products can be rewritten as new octonions.

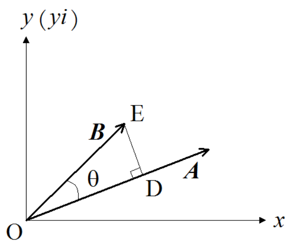


Figure 14.4

Again, we consider two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ on the x-y plane (Figure 14.4). Vector $\boldsymbol{A}$ intercepts with a perpendicular line drawn from the tip E of the vector $\boldsymbol{B}$ at node D. The lengths of the line segments OD and DE are written as $|OD|$ and

$|DE|$, respectively. From the definitions of the inner product and outer product of the vector, i.e.,

$$\boldsymbol{A}\cdot\boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}|\cos\theta, \tag{14.1}$$

$$|\boldsymbol{A}\times\boldsymbol{B}| = |\boldsymbol{A}|\,|\boldsymbol{B}|\sin\theta, \tag{14.2}$$

we have

$$\boldsymbol{A}\cdot\boldsymbol{B} = |\boldsymbol{A}|\,|OD|\,,$$

$$|\boldsymbol{A}\times\boldsymbol{B}| = |\boldsymbol{A}|\,|DE|\,.$$

Rearranging these formulae, we obtain

$$|OD| = \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{A}|}, \tag{14.11}$$

$$|DE| = \frac{|\boldsymbol{A}\times\boldsymbol{B}|}{|\boldsymbol{A}|}. \tag{14.12}$$

The coordinate plane of Figure 14.4 is considered as a complex plane in which the x- and y-axes are real and imaginary axes, respectively. We assume that the complex numbers describing vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are A and B, respectively. As explained in Section 2.2, the coordinate transformation $B\overline{A}/\,|A|$ of the complex number indicates how B is seen from A and we can write

$$\frac{B\overline{A}}{|A|} = |OD| + |DE|\,i.$$

Substituting (14.11) and (14.12) into this equation, we obtain

$$\frac{B\overline{A}}{|A|} = \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{A}|} + \frac{|\boldsymbol{A}\times\boldsymbol{B}|}{|\boldsymbol{A}|}i.$$

Since the magnitude $|\boldsymbol{A}|$ of vector $\boldsymbol{A}$ is the magnitude $|A|$ of the complex number A, it can be written as

$$\frac{B\overline{A}}{|A|} = \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|A|} + \frac{|\boldsymbol{A}\times\boldsymbol{B}|}{|A|}i.$$

Multiplying both sides by $|A|$, this equation becomes

$$B\overline{A} = \boldsymbol{A}\cdot\boldsymbol{B} + |\boldsymbol{A}\times\boldsymbol{B}|\,i. \tag{14.13}$$

Equation (14.13) shows that the inner product $\boldsymbol{A}\cdot\boldsymbol{B}$ of vectors and magnitude $|\boldsymbol{A}\times\boldsymbol{B}|$ of an outer product are performed together when the coordinate transformation $B\overline{A}/\,|A|$ of a complex number is multiplied by $|A|$. That is, the inner product $\boldsymbol{A}\cdot\boldsymbol{B}$ is obtained by multiplying the component of vector $\boldsymbol{B}$ parallel to vector $\boldsymbol{A}$ by $|A|$. The magnitude $|\boldsymbol{A} \times \boldsymbol{B}|$ of an outer product is obtained by multiplying the component of vector $\boldsymbol{B}$ perpendicular to vector $\boldsymbol{A}$ by $|A|$. Thus, the multiplication of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, involving both the inner and outer products, is the multiplication of $|A|$ and the coordinate transformation of $\boldsymbol{B}$ by $\boldsymbol{A}$.

Since (14.13) is computed in a two-dimensional plane, the direction of $|\boldsymbol{A} \times \boldsymbol{B}|\, i$ does not correspond to the direction of the outer product $\boldsymbol{A} \times \boldsymbol{B}$ in three-dimensional space. Thus, we must extend the above theory to three-dimensional space. If the time t of the new octonion $cth + xi + yj + zk$ in the positive world of curved four-dimensional space–time is set to zero, the new octonion in three-dimensional space becomes $xi + yj + zk$. Thus, the new octonions A and B describing vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ in three-dimensional space are written as

$$A = A_x i + A_y j + A_z k, \;\; B = B_x i + B_y j + B_z k. \tag{14.14}$$

In addition, the three-dimensional space component of Hamilton's quaternion in flat four-dimensional space–time is $xi + yj + zk$; thus, (14.14) is realized regardless of the curvature of space. From (14.14), we find

$$\begin{aligned} B\overline{A} &= (B_x i + B_y j + B_z k)(-A_x i - A_y j - A_z k) \\ &= -B_x A_x i^2 - B_x A_y ij - B_x A_z ik \\ &\quad - B_y A_x ji - B_y A_y j^2 - B_y A_z jk \\ &\quad - B_z A_x ki - B_z A_y kj - B_z A_z k^2 \\ &= B_x A_x - B_x A_y k + B_x A_z j \\ &\quad + B_y A_x k + B_y A_y - B_y A_z i \\ &\quad - B_z A_x j + B_z A_y i + B_z A_z \\ &= (A_x B_x + A_y B_y + A_z B_z) \\ &\quad + (A_y B_z - A_z B_y) i + (A_z B_x - A_x B_z) j + (A_x B_y - A_y B_x) k. \end{aligned} \tag{14.15}$$

We know that in terms of the fundamental vectors $\boldsymbol{e_x}$, $\boldsymbol{e_y}$, and $\boldsymbol{e_z}$ the outer product is written as

$$\boldsymbol{A} \times \boldsymbol{B} = (A_y B_z - A_z B_y)\boldsymbol{e_x} + (A_z B_x - A_x B_z)\boldsymbol{e_y} + (A_x B_y - A_y B_x)\boldsymbol{e_z}. \tag{14.10}$$

Equivalently, in teams of the imaginary numbers i, j, and k, we can write

$$\boldsymbol{A} \times \boldsymbol{B} = (A_y B_z - A_z B_y) i + (A_z B_x - A_x B_z) j + (A_x B_y - A_y B_x) k, \tag{14.16}$$

because the fundamental vectors and the imaginary numbers have the same magnitudes and directions. Since the left side of (14.16) is a vector and the right side is a scalar, an equal symbol cannot be used. However, since the semantic of both sides is the same, we adopt this notation hereafter.

From (14.15), (14.16), and the coordinate components expression of the inner vector product, i.e.,

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z, \tag{14.6}$$

we can write

$$B\overline{A} = \boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A}\times\boldsymbol{B}. \tag{14.17}$$

Equation (14.17) extends the formula in the two-dimensional plane, i.e.,

$$B\overline{A} = \boldsymbol{A}\cdot\boldsymbol{B} + |\boldsymbol{A}\times\boldsymbol{B}|\, i, \tag{14.13}$$

to three-dimensional space. From (14.17), we see that $B\overline{A}$ involves the simultaneous calculation of the inner and outer vector products in three-dimensional space. Furthermore, the multiplication of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ containing both products is the multiplication of $|A|$ and the coordinate transformation of $\boldsymbol{B}$ by $\boldsymbol{A}$. In other words, the multiplication of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ indicates the multiplication of $\boldsymbol{B}$ viewed from $\boldsymbol{A}$ by $|A|$ in three-dimensional space.

We now present an alternative proof that the inner and outer products of two vectors are coordinate transformations. The inner product $\boldsymbol{A}\cdot\boldsymbol{B}$ multiplies $|\boldsymbol{A}|$ with the component of $\boldsymbol{B}$ parallel to vector $\boldsymbol{A}$. The magnitude $|\boldsymbol{A}\times\boldsymbol{B}|$ of an outer product denotes the multiplication of $|\boldsymbol{A}|$ with the component of $\boldsymbol{B}$ perpendicular to vector $\boldsymbol{A}$. The magnitude $|\boldsymbol{B}|$ of $\boldsymbol{B}$ remains unchanged after a coordinate transformation. Thus, by Pythagoras' theorem, we can write

$$\frac{|\boldsymbol{A}\cdot\boldsymbol{B}|^2}{|\boldsymbol{A}|^2} + \frac{|\boldsymbol{A}\times\boldsymbol{B}|^2}{|\boldsymbol{A}|^2} = |\boldsymbol{B}|^2,$$

$$|\boldsymbol{A}\cdot\boldsymbol{B}|^2 + |\boldsymbol{A}\times\boldsymbol{B}|^2 = |\boldsymbol{B}|^2\,|\boldsymbol{A}|^2. \tag{14.18}$$

If an inner and outer products, expressed in terms of their coordinate components, can be proven to satisfy (14.18), then the inner and outer vector products are coordinate transformations. Substituting the following two equations

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z, \tag{14.9}$$

$$\boldsymbol{A}\times\boldsymbol{B} = (A_yB_z - A_zB_y)i + (A_zB_x - A_xB_z)j + (A_xB_y - A_yB_x)k \tag{14.16}$$

into the left-hand side of (14.18), we find that

$$\begin{aligned}|\boldsymbol{A}\cdot\boldsymbol{B}|^2 + |\boldsymbol{A}\times\boldsymbol{B}|^2 &= (A_xB_x + A_yB_y + A_zB_z)^2 \\ &\quad + [(A_yB_z - A_zB_y)i + (A_zB_x - A_xB_z)j + (A_xB_y - A_yB_x)k] \\ &\quad \times [-(A_yB_z - A_zB_y)i - (A_zB_x - A_xB_z)j - (A_xB_y - A_yB_x)k].\end{aligned}$$

Since the calculations are lengthy, they are omitted here. If the algebra is followed through, we obtain

$$\begin{aligned} |\boldsymbol{A}\cdot\boldsymbol{B}|^2 + |\boldsymbol{A}\times\boldsymbol{B}|^2 &= (B_x^2 + B_y^2 + B_z^2)(A_x^2 + A_y^2 + A_z^2) \\ &= |\boldsymbol{B}|^2\,|\boldsymbol{A}|^2\,. \end{aligned}$$

14.3 New octonions and direction of outer products of vectors

Section 14.1 explained that, in mathematics and physics texts, the outer product $\boldsymbol{A}\times\boldsymbol{B}$ of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is assumed perpendicular to the plane described by the vectors without providing proof. The definition is provided without a rigorous rationale. In this section, using

$$\begin{aligned} B\overline{A} = {} & (A_xB_x + A_yB_y + A_zB_z) \\ & + (A_yB_z - A_zB_y)i + (A_zB_x - A_xB_z)j + (A_xB_y - A_yB_x)k \end{aligned} \tag{14.15}$$

and

$$B\overline{A} = \boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A}\times\boldsymbol{B}, \tag{14.17}$$

we prove that if a vector is rewritten by the new octonion, the outer product is inevitably oriented perpendicular to the plane described by vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.

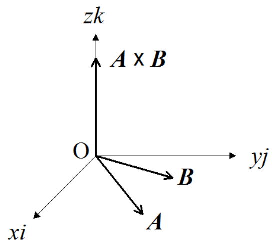


Figure 14.5

As shown in Figure 14.5, if the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ lie on the xi-yj plane, since $A_z = B_z = 0$, (14.15) becomes

$$B\overline{A} = (A_xB_x + A_yB_y) + (A_xB_y - A_yB_x)k.$$

By this formula and (14.17), we find that

$$\boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A}\times\boldsymbol{B} = (A_xB_x + A_yB_y) + (A_xB_y - A_yB_x)k.$$

Since the scalar $\boldsymbol{A}\cdot\boldsymbol{B}$ is directionless but the vector $\boldsymbol{A}\times\boldsymbol{B}$ has a direction, we can write

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_x B_x + A_y B_y, \tag{14.19}$$

$$\boldsymbol{A}\times\boldsymbol{B} = (A_x B_y - A_y B_x)k. \tag{14.20}$$

Since in (14.20), the imaginary number k lies along the zk-axis, the outer product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ on the xi-yj plane orients in the zk-direction. This statement is a mathematical proof of the direction of the outer product $\boldsymbol{A}\times\boldsymbol{B}$. It cannot be found in standard mathematics and physics texts. Vector mathematics is successful only if the direction of the outer product $\boldsymbol{A}\times\boldsymbol{B}$ is set perpendicular to the plane described by vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. Above, this universal fact has been mathematically proven.

14.4 Calculation of three-vector products by the new octonion

The previous analyses were performed on two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. Multiplications of three vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are called triple products. The important three-vector products are the triple scalar product $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$ and the triple vector product $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})$. This section shows that triple products can be rewritten in terms of the new octonion.

In Hamilton's notation, the components $(x,\ y,\ z)$ of the vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are written as

$$(A_x,\ A_y,\ A_z),\ (B_x,\ B_y,\ B_z),\ (C_x,\ C_y,\ C_z).$$

In terms of the fundamental vectors of magnitude 1 along the x-, y-, and z-axes, namely, $\boldsymbol{e_x}$, $\boldsymbol{e_y}$, and $\boldsymbol{e_z}$, the vectors become

$$\boldsymbol{A} = A_x\boldsymbol{e_x} + A_y\boldsymbol{e_y} + A_z\boldsymbol{e_z},$$
$$\boldsymbol{B} = B_x\boldsymbol{e_x} + B_y\boldsymbol{e_y} + B_z\boldsymbol{e_z},$$
$$\boldsymbol{C} = C_x\boldsymbol{e_x} + C_y\boldsymbol{e_y} + C_z\boldsymbol{e_z}.$$

The calculations can be found in any standard text on vectors, and are hence omitted. From the properties of the inner and outer products of the fundamental vectors $\boldsymbol{e_x}$, $\boldsymbol{e_y}$, and $\boldsymbol{e_z}$, given by Equations (14.4), (14.5), (14.7), (14.8), and (14.9), the triple scalar product is obtained as

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = A_x(B_y C_z - B_z C_y) + A_y(B_z C_x - B_x C_z) + A_z(B_x C_y - B_y C_x), \tag{14.21}$$

and the triple vector product is

$$\begin{aligned}\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = {} & [(A_yC_y + A_zC_z)B_x - (A_yB_y + A_zB_z)C_x]\boldsymbol{e_x} \\ & + [(A_zC_z + A_xC_x)B_y - (A_zB_z + A_xB_x)C_y]\boldsymbol{e_y} \\ & + [(A_xC_x + A_yC_y)B_z - (A_xB_x + A_yB_y)C_z]\boldsymbol{e_z}. \end{aligned} \tag{14.22}$$

Next, we express the triple products as new octonions. The new octonions describing vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are A, B, and C, respectively. From the expression

$$B\overline{A} = \boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A}\times\boldsymbol{B} \tag{14.17}$$

obtained in Section 14.2, we can write

$$C\overline{B} = \boldsymbol{B}\cdot\boldsymbol{C} + \boldsymbol{B}\times\boldsymbol{C}, \tag{14.23}$$

$$B\overline{C} = \boldsymbol{C}\cdot\boldsymbol{B} + \boldsymbol{C}\times\boldsymbol{B}. \tag{14.24}$$

From the definition of the outer product, we have

$$\boldsymbol{C}\times\boldsymbol{B} = -\boldsymbol{B}\times\boldsymbol{C}.$$

Thus, (14.24) becomes

$$B\overline{C} = \boldsymbol{C}\cdot\boldsymbol{B} - \boldsymbol{B}\times\boldsymbol{C}. \tag{14.25}$$

Since $\boldsymbol{B}\cdot\boldsymbol{C} = \boldsymbol{C}\cdot\boldsymbol{B}$, subtracting (14.25) from (14.23) yields

$$C\overline{B} - B\overline{C} = 2(\boldsymbol{B}\times\boldsymbol{C}).$$

Thus, we can write

$$\boldsymbol{B}\times\boldsymbol{C} = (C\overline{B} - B\overline{C})/2. \tag{14.26}$$

In addition, adding (14.26) to (14.23), we obtain

$$C\overline{B} + \boldsymbol{B}\times\boldsymbol{C} = \boldsymbol{B}\cdot\boldsymbol{C} + \boldsymbol{B}\times\boldsymbol{C} + (C\overline{B} - B\overline{C})/2.$$

Thus, we can write

$$\boldsymbol{B}\cdot\boldsymbol{C} = (C\overline{B} + B\overline{C})/2. \tag{14.27}$$

Equations (14.26) and (14.27) were incorporated into Theorem 28 in Section 13.3.

Since the purpose of this section is to formulate the triple products as new octonions, we verify the results in advance. For simplicity, we set $\boldsymbol{D} = \boldsymbol{B}\times\boldsymbol{C}$, so that (14.26) becomes

$$\boldsymbol{D} = (C\overline{B} - B\overline{C})/2.$$

Since the new octonion D is considered identical to vector $\boldsymbol{D}$, we also define

$$D = (C\overline{B} - B\overline{C})/2. \tag{14.28}$$

From the equation

$$B\overline{A} = \boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{B} \tag{14.17}$$

obtained in Section 14.2, we can write

$$D\overline{A} = \boldsymbol{A}\cdot\boldsymbol{D} + \boldsymbol{A} \times \boldsymbol{D}.$$

In terms of $\boldsymbol{D} = \boldsymbol{B} \times \boldsymbol{C}$ and (14.28), this equation becomes

$$(C\overline{B} - B\overline{C})\overline{A}/2 = \boldsymbol{A}\cdot(\boldsymbol{B} \times \boldsymbol{C}) + \boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}). \tag{14.29}$$

Equation (14.29) relates the triple products to the new octonion, and is used as follows.

(1) Construct the new octonions A, B, and C from the coordinate components of vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, respectively.
(2) Calculate $(C\overline{B} - B\overline{C})\overline{A}/2$.
(3) The real-number component of $(C\overline{B} - B\overline{C})\overline{A}/2$ is the triple scalar product $\boldsymbol{A}\cdot(\boldsymbol{B} \times \boldsymbol{C})$, while the imaginary component is the triple vector product $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$.

The inner and outer products of vectors are computed together in (14.29), as established earlier for two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. For convenience, areas and volumes are calculated from outer products and from triple scalar products, respectively. However, because the vector multiplications are essentially coordinate transformations, the inner and outer products appear together in the new octonion.

We now demonstrate that (14.29) is correct. Substituting

$$A = A_x i + A_y j + A_z k,$$
$$B = B_x i + B_y j + B_z k,$$
$$C = C_x i + C_y j + C_z k$$

for $(C\overline{B} - B\overline{C})\overline{A}/2$, we find

$$\begin{aligned}
&(C\overline{B} - B\overline{C})\overline{A}/2 \\
&= [(C_x i + C_y j + C_z k)(-B_x i - B_y j - B_z k) - (B_x i + B_y j + B_z k)(-C_x i - C_y j - C_z k)] \\
&\quad \times (-A_x i - A_y j - A_z k)/2 \\
&= A_x(B_y C_z - B_z C_y) + A_y(B_z C_x - B_x C_z) + A_z(B_x C_y - B_y C_x)
\end{aligned}$$

$$+ [(A_y C_y + A_z C_z)B_x - (A_y B_y + A_z B_z)C_x]i$$
$$+ [(A_z C_z + A_x C_x)B_y - (A_z B_z + A_x B_x)C_y]j$$
$$+ [(A_x C_x + A_y C_y)B_z - (A_x B_x + A_y B_y)C_z]k.$$

The calculations are lengthy, and are therefore omitted. Since

$$\boldsymbol{A}\cdot(\boldsymbol{B} \times \boldsymbol{C}) = A_x(B_y C_z - B_z C_y) + A_y(B_z C_x - B_x C_z) + A_z(B_x C_y - B_y C_x), \tag{14.21}$$

$$\begin{aligned}\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = {} & [(A_y C_y + A_z C_z)B_x - (A_y B_y + A_z B_z)C_x]\boldsymbol{e_x} \\ & + [(A_z C_z + A_x C_x)B_y - (A_z B_z + A_x B_x)C_y]\,\boldsymbol{e_y} \\ & + [(A_x C_x + A_y C_y)B_z - (A_x B_x + A_y B_y)C_z]\boldsymbol{e_z}\end{aligned} \tag{14.22}$$

as previously discussed, the above equation is simplified to

$$(C\overline{B} - B\overline{C})\overline{A}/2 = \boldsymbol{A}\cdot(\boldsymbol{B} \times \boldsymbol{C}) + \boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}).$$

However, since we consider that the fundamental vectors, $\boldsymbol{e_x}$, $\boldsymbol{e_y}$, and $\boldsymbol{e_z}$, and imaginary numbers, i, j, and k, express the same content, we also regard them as mutually interchangeable.

14.5 Calculation of four-vector products by the new octonion

By the method used to obtain the triple product, we can relate the quadruple product to the new octonion in the case of four vectors, $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$. The sum of the quadruple scalar and vector products is

$$(\boldsymbol{A} \times \boldsymbol{B})\cdot(\boldsymbol{C} \times \boldsymbol{D}) + (\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D}).$$

From the formula

$$Q\overline{P} = \boldsymbol{P}\cdot\boldsymbol{Q} + \boldsymbol{P} \times \boldsymbol{Q},$$

we find

$$\{\boldsymbol{C} \times \boldsymbol{D}\}\left\{\overline{\boldsymbol{A} \times \boldsymbol{B}}\right\} = (\boldsymbol{A} \times \boldsymbol{B})\cdot(\boldsymbol{C} \times \boldsymbol{D}) + (\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D}). \tag{14.30}$$

Here, $\{\boldsymbol{C} \times \boldsymbol{D}\}$ is a new octonion describing the vector $\boldsymbol{C} \times \boldsymbol{D}$, and $\left\{\overline{\boldsymbol{A} \times \boldsymbol{B}}\right\}$ is the new octonion conjugate of the new octonion $\{\boldsymbol{A} \times \boldsymbol{B}\}$ describing the vector $\boldsymbol{A} \times \boldsymbol{B}$. From the formula

$$\{\boldsymbol{B} \times \boldsymbol{C}\} = (C\overline{B} - B\overline{C})/2, \tag{14.26}$$

we can write

$$\{\boldsymbol{A}\times\boldsymbol{B}\} = (B\overline{A} - A\overline{B})/2,$$
$$\{\boldsymbol{C}\times\boldsymbol{D}\} = (D\overline{C} - C\overline{D})/2. \tag{14.31}$$

In addition, given that $\overline{PQ} = \overline{Q}\,\overline{P}$, the new octonion conjugate of the new octonion $B\overline{A} - A\overline{B}$ is $A\overline{B} - B\overline{A}$. Thus, we obtain

$$\{\overline{\boldsymbol{A}\times\boldsymbol{B}}\} = (A\overline{B} - B\overline{A})/2. \tag{14.32}$$

Substituting (14.31) and (14.32) into the left-hand side of (14.30), we have

$$(D\overline{C} - C\overline{D})(A\overline{B} - B\overline{A})/4 = (\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D}) + (\boldsymbol{A}\times\boldsymbol{B})\times(\boldsymbol{C}\times\boldsymbol{D}).$$

This equation relates the quadruple product to the new octonion.

14.6 New octonion and the four-dimensional vector

The previous analyses were conducted in three-dimensional space with the time dimension set to zero. This section examines the inner and outer products of vectors in the whole curved four-dimensional space–time, including both positive and negative worlds.

If we assume that the coordinates $(cth,\ xi,\ yj,\ zk)$ of two four-dimensional vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ in the positive world are

$$(A_th,\ A_xi,\ A_yj,\ A_zk),\ (B_th,\ B_xi,\ B_yj,\ B_zk),$$

the new octonions A and B describing vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are

$$A = A_th + A_xi + A_yj + A_zk,$$
$$B = B_th + B_xi + B_yj + B_zk.$$

If we assume that the formula

$$B\overline{A} = \boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A}\times\boldsymbol{B} \tag{14.17}$$

is also realized in four-dimensional space–time, we find that

$$\begin{aligned}B\overline{A} &= (B_th + B_xi + B_yj + B_zk)(A_th - A_xi - A_yj - A_zk)\\ &= B_tA_th^2 - B_tA_xhi - B_tA_yhj - B_tA_zhk\\ &\quad + B_xA_thi - B_xA_xi^2 - B_xA_yij - B_xA_zik\\ &\quad + B_yA_thj - B_yA_xji - B_yA_yj^2 - B_yA_zjk\\ &\quad + B_zA_thk - B_zA_xki - B_zA_ykj - B_zA_zk^2\end{aligned}$$

$$
\begin{aligned}
&= -B_tA_t - B_tA_xhi - B_tA_yhj - B_tA_zhk \\
&\quad + B_xA_thi + B_xA_x - B_xA_yk + B_xA_zj \\
&\quad + B_yA_thj + B_yA_xk + B_yA_y - B_yA_zi \\
&\quad + B_zA_thk - B_zA_xj + B_zA_yi + B_zA_z \\
&= (-B_tA_t + B_xA_x + B_yA_y + B_zA_z) \\
&\quad + (B_xA_t - B_tA_x)hi + (B_yA_t - B_tA_y)hj + (B_zA_t - B_tA_z)hk \\
&\quad + (B_zA_y - B_yA_z)i + (B_xA_z - B_zA_x)j + (B_yA_x - B_xA_y)k \\
&= (-A_tB_t + A_xB_x + A_yB_y + A_zB_z) \\
&\quad + (A_tB_x - A_xB_t)hi + (A_tB_y - A_yB_t)hj + (A_tB_z - A_zB_t)hk \\
&\quad + (A_yB_z - A_zB_y)i + (A_zB_x - A_xB_z)j + (A_xB_y - A_yB_x)k.
\end{aligned}
$$

Thus, we obtain

$$\boldsymbol{A}\cdot\boldsymbol{B} = -A_tB_t + A_xB_x + A_yB_y + A_zB_z, \tag{14.33}$$

$$
\begin{aligned}
\boldsymbol{A}\times\boldsymbol{B} &= (A_tB_x - A_xB_t)hi + (A_tB_y - A_yB_t)hj + (A_tB_z - A_zB_t)hk \\
&\quad + (A_yB_z - A_zB_y)i + (A_zB_x - A_xB_z)j + (A_xB_y - A_yB_x)k.
\end{aligned} \tag{14.34}
$$

Equations (14.33) and (14.34) are expressed in terms of the coordinate components of the inner and outer products of the four-dimensional vectors. If the time components A_t and B_t in these formulae are set to zero, these equations reduce to

$$
\begin{aligned}
\boldsymbol{A}\cdot\boldsymbol{B} &= A_xB_x + A_yB_y + A_zB_z, \\
\boldsymbol{A}\times\boldsymbol{B} &= (A_yB_z - A_zB_y)i + (A_zB_x - A_xB_z)j + (A_xB_y - A_yB_x)k.
\end{aligned}
$$

These are identical to Equations (14.6) and (14.10), i.e.,

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z, \tag{14.6}$$

$$\boldsymbol{A}\times\boldsymbol{B} = (A_yB_z - A_zB_y)\boldsymbol{e_x} + (A_zB_x - A_xB_z)\boldsymbol{e_y} + (A_xB_y - A_yB_x)\boldsymbol{e_z}, \tag{14.10}$$

describing the inner and outer products of two three-dimensional vectors. However, (14.10) is expressed in terms of the fundamental vectors, $\boldsymbol{e_x}$, $\boldsymbol{e_y}$, and $\boldsymbol{e_z}$, rather than the imaginary numbers i, j, and k.

Since (14.34) is difficult to memorize, we rewrite it as a determinant. The determinant of a matrix is found by multiplying the right-diagonal elements (from top to bottom rows) and adding the components, then multiplying the left-diagonal elements (from top to bottom rows) and subtracting the components from the positive components. Below, this process is demonstrated by a pair of simple examples.

$$\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} = a_1a_4 - a_2a_3,$$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k.$$

In terms of the determinants of its components, (14.34) becomes

$$\begin{aligned} \boldsymbol{A} \times \boldsymbol{B} = & \begin{vmatrix} A_t & A_x \\ B_t & B_x \end{vmatrix} hi + \begin{vmatrix} A_t & A_y \\ B_t & B_y \end{vmatrix} hj + \begin{vmatrix} A_t & A_z \\ B_t & B_z \end{vmatrix} hk \\ & + \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} i + \begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} j + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} k. \end{aligned} \tag{14.35}$$

Similarly, the i, j, and k components of (14.35) are rewritten as

$$\begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

In this form, the determinant does not show the relationships between the i, j, and k components and the hi, hj, and hk components; these become clarified when the determinant is rewritten as (14.35).

In new octonion algebra, the real number component expresses the time component in the negative world (see Section 11.4). Thus, from

$$\boldsymbol{A}\cdot\boldsymbol{B} = -A_tB_t + A_xB_x + A_yB_y + A_zB_z, \tag{14.33}$$

the inner product of two vectors in four-dimensional space–time denotes the time component in the negative world. Because this time component is inaccessible to us, we observe only the outer product when multiplying two vectors. This partly explains why we observe the inner and outer vector products as different.

In addition, because the outer product

$$\begin{aligned} \boldsymbol{A} \times \boldsymbol{B} = & (A_tB_x - A_xB_t)hi + (A_tB_y - A_yB_t)hj + (A_tB_z - A_zB_t)hk \\ & + (A_yB_z - A_zB_y)i + (A_zB_x - A_xB_z)j + (A_xB_y - A_yB_x)k \end{aligned} \tag{14.34}$$

contains hi, hj, and hk components, it denotes the spatial component in both positive and negative worlds. Like the inner product, we cannot observe the spatial component in the negative world. Thus, setting the hi, hj, and hk components of (14.34) to zero, we retrieve the definition of the outer product until now.

Equations (14.33) and (14.34) correctly describe the inner and outer products, respectively, in curved four-dimensional space–time. Furthermore, replacing the imaginary number h of the new octonion by 1, we restore Hamilton's quaternion, which describes the mathematics of flat four-dimensional space–time. Thus, in flat four-dimensional space–time, (14.33) and (14.34) can be rewritten as follows:

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_t B_t + A_x B_x + A_y B_y + A_z B_z, \tag{14.35}$$

$$\begin{aligned}\boldsymbol{A}\times\boldsymbol{B} = &\begin{vmatrix} A_t & A_x \\ B_t & B_x \end{vmatrix} i + \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} i \\ &+ \begin{vmatrix} A_t & A_y \\ B_t & B_y \end{vmatrix} j + \begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} j \\ &+ \begin{vmatrix} A_t & A_z \\ B_t & B_z \end{vmatrix} k + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} k.\end{aligned} \tag{14.36}$$

Since no negative world is recognized in flat four-dimensional space–time, Equations (14.35) and (14.36) express the temporal and spatial components, respectively, in the positive world. While these formulae are easily understood, they do not acknowledge relativity theory in their realized world. Assuming that relativity theory is correct, (14.33) and (14.34) correctly formulate the inner and outer vector products in four-dimensional space–time.

The correctness of Equations (14.33)−−(14.36) is easily proved by demonstrating that the equations satisfy

$$|\boldsymbol{A}\cdot\boldsymbol{B}|^2 + |\boldsymbol{A}\times\boldsymbol{B}|^2 = |\boldsymbol{B}|^2\,|\boldsymbol{A}|^2$$

derived in Section 14.2. Calculations of this proof are tedious, and are hence omitted.

In this chapter, we have demonstrated that vector calculations can be rewritten in terms of the new octonion. Since the concept of the vector is now established in physics, rewriting physics in terms of the new octonion remains a major challenge. Successors of Hamilton who discovered the quaternion failed in their attempts to rewrite Maxwell's electromagnetic equations, because their new mathematics is applicable only to flat space–time.

Since standard vector analysis is the mathematics of flat three-dimensional space, they must be rewritten as the mathematics of curved four-dimensional space–time using Equations (14.33) and (14.34) which describe the inner and outer products of the new octonion.

14.7 New octonion and rotation vector

This section shows how the rotation vector, which plays a central role in physics, may be rewritten as a new octonion. This is an example of rewriting physical laws by the new octonin. First, we explain what is meant by a rotation.

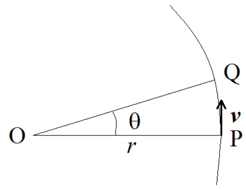


Figure 14.6

Suppose that a point mass A moves along the circumference of the circle of radius r from point P to point Q through an angle θ (theta). The length l of the arc described by the movement of the point mass is

$$l = r\theta. \tag{14.37}$$

The reader should note that θ is expressed in radians, not degrees. A radian is the length of arc equivalent to unit radius (1 radian $\sim 57.3°$). If a point mass travels through an infinitesimal angle $d\theta$ during infinitesimal time dt, the infinitesimal distance traveled is the infinitesimal arc length dl. Therefore, the infinitesimal limit of (14.37) is

$$dl = rd\theta.$$

Dividing both sides of this formula by the infinitesimal time dt, we obtain

$$\frac{dl}{dt} = r\frac{d\theta}{dt}, \tag{14.38}$$

where dl/dt is the velocity v of the rotating point mass at point P. This direction is tangential to the circumference. In addition, since $d\theta/dt$ describes the temporal change in angle, it is defined as the angular velocity ω (omega). Therefore, (14.38) becomes

$$v = r\omega.$$

To ensure agreement with Equation (14.40), derived later, this expression is rewritten as

$$v = \omega r. \tag{14.39}$$

Japanese physics students are not introduced to vectors until they enter university. It is the rotation vector that is frequently bewildering to students encountering vectors for the first time. Below, we give a detailed description of the rotation vector.

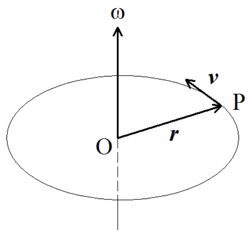


Figure 14.7

Suppose that point P travels with velocity $\boldsymbol{v}$ around the circumference of a circle of radius r centered at O. Then, vectors $\boldsymbol{r}$ and $\boldsymbol{v}$ are related to the angular velocity vector $\boldsymbol{\omega}$, through the outer vector product:

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}.$$

The direction of $\boldsymbol{\omega}$ is perpendicular to the plane described by the vectors $\boldsymbol{r}$ and $\boldsymbol{v}$. Since $\boldsymbol{\omega}$ is perpendicular to the rotational plane, its direction is that of the rotation axis. Although the rotation axis may be oriented up or down, its direction is conventionally defined as that moved by a right-handed screw turning in the direction of $\boldsymbol{v}$. In Figure 14.7, this direction is upward.

The definition of the angular velocity vector $\boldsymbol{\omega}$ is bewildering to students because, although rotation is planar, the angular velocity vector is directed perpendicular, and therefore contradictory to the direction of angular movement. By contrast, force and velocity vectors are oriented parallel to the force and velocity. However, the rotational and actual directions of the angular velocity vector $\boldsymbol{\omega}$ are not the same, for reasons that are mentioned, but not discussed, in standard physics texts. This case is analogous to the outer product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ being assigned, but not proved, perpendicularly oriented to the plane described by the vectors. Establishing a rule because it is convenient is neither rigorous nor logical.

In Section 14.3, we used the new octonion to prove that the outer product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is perpendicular to the plane described by the vectors. Similarly, the new octonion can prove that the direction of the angular velocity vector $\boldsymbol{\omega}$ is perpendicular to the plane described by the vectors $\boldsymbol{r}$ and $\boldsymbol{v}$. In (14.39), we assume that the angular velocity ω is an unknown function of two vectors $\boldsymbol{r}$ and $\boldsymbol{v}$. Since $\boldsymbol{r}$

and $\boldsymbol{v}$ are vectors while ω is not, (14.39) becomes

$$\boldsymbol{v} = \omega \boldsymbol{r}. \tag{14.40}$$

In terms of the coordinate components, the new octonions of $\boldsymbol{r}$ and $\boldsymbol{v}$ can be written as

$$v = v_x i + v_y j + v_z k,$$
$$r = xi + yj + zk,$$

by which (14.40) becomes

$$(v_x i + v_y j + v_z k) = \omega(xi + yj + zk).$$

This formula relates two new octonions by a rotation. For ease of calculation, this expression can be reverted to the new octonion itself, giving

$$v = \omega r. \tag{14.41}$$

Although (14.41) appears identical to (14.39), the latter is formulated as a real number while (14.41) is formulated in new octonions. Furthermore, ω in (14.41) is an unknown function that relates vectors $\boldsymbol{r}$ and $\boldsymbol{v}$. Multiplying both sides of (14.41) by a right-handed $\overline{r}$, we obtain

$$\begin{aligned} v\overline{r} &= \omega r \overline{r} \\ &= \omega \left|r\right|^2. \end{aligned}$$

For $r \neq 0$, we find that

$$\omega = \frac{v\overline{r}}{\left|r\right|^2}, \tag{14.42}$$

which expresses the rotation velocity ω in terms of the new octonion.

Using this equation, we now determine the direction of ω. From the relationship between the new octonion and the vector presented in Section 14.2, i.e.,

$$B\overline{A} = \boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{B},$$

we can write

$$v\overline{r} = \boldsymbol{r}\cdot\boldsymbol{v} + \boldsymbol{r} \times \boldsymbol{v}.$$

Thus, (14.42) becomes

$$\omega = \frac{1}{\left|r\right|^2}(\boldsymbol{r}\cdot\boldsymbol{v} + \boldsymbol{r} \times \boldsymbol{v}). \tag{14.43}$$

When the radius r is constant, i.e., when the path is truly circular, the angle made by $\boldsymbol{r}$ and $\boldsymbol{v}$ is a right angle (i.e., $\pi/2$ radian). Thus, from the definition of an inner product, we have

$$\boldsymbol{r}\cdot\boldsymbol{v} = |\boldsymbol{r}|\,|\boldsymbol{v}|\cos\frac{\pi}{2} = 0.$$

Therefore, if the radius r is constant, (14.43) reduces to

$$\omega = \frac{1}{|r|^2}(\boldsymbol{r}\times\boldsymbol{v}).$$

Since ω is a new octonion, it expresses the same content as the vector $\boldsymbol{\omega}$. Hence $\boldsymbol{r}$, $\boldsymbol{v}$, and the angular velocity vector $\boldsymbol{\omega}$ are related by

$$\boldsymbol{\omega} = \frac{1}{|r|^2}(\boldsymbol{r}\times\boldsymbol{v}). \tag{14.44}$$

Together with the formula of the outer vector product, (14.44) shows that the angular velocity vector $\boldsymbol{\omega}$ is perpendicular to the plane described by $\boldsymbol{r}$ and $\boldsymbol{v}$. Equation (14.44) provides mathematical proof of the perpendicularity between $\boldsymbol{v}$ and angular velocity vector $\boldsymbol{\omega}$. In addition, when point P travels an ellipsoid path, the inner product $\boldsymbol{r}\cdot\boldsymbol{v}$ is non-zero. Thus, the angular velocity vector $\boldsymbol{\omega}$ possesses a temporal component $\boldsymbol{r}\cdot\boldsymbol{v}$ in the negative world, as well as a spatial component perpendicular to the rotational plane.

When $\boldsymbol{r}$ and $\boldsymbol{v}$ are parallel to the direction of motion, $\boldsymbol{r}\times\boldsymbol{v} = 0$. In this case, the equation

$$\omega = \frac{1}{|r|^2}(\boldsymbol{r}\cdot\boldsymbol{v} + \boldsymbol{r}\times\boldsymbol{v}) \tag{14.43}$$

reduces to

$$\begin{aligned}\omega &= \frac{1}{|r|^2}\boldsymbol{r}\cdot\boldsymbol{v}\\ &= \frac{1}{|r|^2}\,|r|\,|v|\\ &= \frac{|v|}{|r|}\\ &= \frac{|r|\,/t}{|r|}\\ &= \frac{1}{t},\end{aligned}$$

where t denotes time. Then, we can write

$$\begin{aligned}v &= \omega r\\ &= \frac{r}{t}.\end{aligned}$$

This equation relates the velocity to the distance traveled in linear motion.

From the above results, the equation

$$\omega = \frac{1}{|r|^2}(\boldsymbol{r}\cdot\boldsymbol{v} + \boldsymbol{r} \times \boldsymbol{v}) \tag{14.43}$$

expresses the general relationship between the position vector $\boldsymbol{r}$, rate vector $\boldsymbol{v}$, and angular velocity ω in three-dimensional space. Equation (14.43) is varid only when calculated by the new octonion.

15

New Octonions and Tensors

15.1 Reasons why tensors were made

A student is not introduced to tensors until he/she enters the third year of the science and engineering degree at a University in Japan. Thus, many ordinary readers and college students will be unfamiliar with the tensor, although tensors are an essential component of higher science and engineering learning. In the engineering field, they are used to calculate tensions in objects. Einstein's general relativity theory of gravity is also written in a tensor form.

Despite its importance in present-day science and engineering, the tensor remains obscure to most students, most likely for the following reasons:
(1) Few books provide a rationale for developing the tensor.
(2) The tensor cannot be matched to a concrete image.

Most tensor texts begin by defining the tensor and then presenting examples and tensor formulas. As a result, the necessity of the tensor is neglected. When the imaginary number is introduced at high school, its rationale is explained; the square of the imaginary number i equals -1. We can intuitively understand that trigonometric functions describe the relationships between the angles and side lengths of triangles. Quantities possessing both magnitude and direction are easily represented by vectors. However, by comparison, the tensor lacks an obvious rationale, and therefore seems mysterious and obscure. This obscurity is partly responsible for why tensors are so difficult to understand.

Tensors are further complicated by their lack of connection to any concrete image. The imaginary number described previously can be plotted as a point on a complex plane, while a trigonometric function is easily visualized on a triangle. A vector is denoted by an arrow. Every number can be visualized as an image.

Individuals who memorize sequences of hundreds of irregular numbers, such as the circular constant π, do so by relating numbers to concrete images and memorable tale. Since the tensor can neither be plotted on a figure nor be interpreted as an image, it cannot be readily understood and memorized. To redress this issue, this section explains why the tensor was postulated and why it cannot be expressed by a diagram. This explanation is essential for understanding the content of the following section, 15.2.

The wise student who has studied vectors in high school might recognize that two vectors cannot be divided. At least, nobody has proposed the division of a vector. Real numbers, imaginary numbers, and trigonometric functions are all divisible. As vectors are indivisible, they cannot be manipulated by a full set of mathematical operations. The tensor was proposed to accommodate division in vector quantities.

First, we consider real number division. If two real numbers x and y are related by

$$y = ax$$

and $x \neq 0$, then deviding both sides of the equation by x, we obtain

$$a = \frac{y}{x}.$$

Furthermore, if $x = 1$ and $y = 2$, we have

$$a = \frac{2}{1} = 2.$$

Thus, $y = ax$ becomes

$$y = 2x.$$

Inserting $x = 3$ into this equation gives

$$y = 6.$$

Therefore, in the equation $y = 2x$, numbers 1 and 2 and numbers 3 and 6 are related in the same way (through the constant 2).

Similar relationships exist between two vectors. We assume the following relationship between vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, where T denotes a tensor.

$$\boldsymbol{B} = T\boldsymbol{A}. \tag{15.1}$$

At present, the contents of T are unknown. As explained in Section 14.1, if vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are expressed in terms of their coordinate components $(A_x,\ A_y,\ A_z)$ and $(B_x,\ B_y,\ B_z)$, and their fundamental vectors $\boldsymbol{e_x},\ \boldsymbol{e_y}$, and $\boldsymbol{e_z}$, (15.1) becomes

$$B_x\boldsymbol{e_x} + B_y\boldsymbol{e_y} + B_z\boldsymbol{e_z} = T(A_x\boldsymbol{e_x} + A_y\boldsymbol{e_y} + A_z\boldsymbol{e_z}). \tag{15.2}$$

Assuming that both sides of (15.2) are divisible by $A_x\boldsymbol{e_x} + A_y\boldsymbol{e_y} + A_z\boldsymbol{e_z}$, we can write

$$T = \frac{B_x\boldsymbol{e_x} + B_y\boldsymbol{e_y} + B_z\boldsymbol{e_z}}{A_x\boldsymbol{e_x} + A_y\boldsymbol{e_y} + A_z\boldsymbol{e_z}}. \tag{15.3}$$

To eliminate the fundamental vectors $\boldsymbol{e_x}$, $\boldsymbol{e_y}$, and $\boldsymbol{e_z}$ from the denominator of (15.3), we multiply the numerator and denominator by some vector. Since multiplication of two vectors yields an inner or an outer product, the explicit form of the multiplication is unknown, and hence, we retain (15.3) in the above format. Multiplying (15.3) by vector $\boldsymbol{C}(C_x,\ C_y,\ C_z)$, vector $\boldsymbol{D}$ is obtained as

$$\begin{aligned}\boldsymbol{D} &= T\boldsymbol{C}\\ &= \frac{B_x\boldsymbol{e_x} + B_y\boldsymbol{e_y} + B_z\boldsymbol{e_z}}{A_x\boldsymbol{e_x} + A_y\boldsymbol{e_y} + A_z\boldsymbol{e_z}} \times (C_x\boldsymbol{e_x} + C_y\boldsymbol{e_y} + C_z\boldsymbol{e_z}).\end{aligned}$$

Even in this format, whether the result of multiplication of the fundamental vectors $\boldsymbol{e_x}$, $\boldsymbol{e_y}$, and $\boldsymbol{e_z}$ is an inner or outer product is indeterminable and further calculations are impossible. As the vector $\boldsymbol{D}$ is incalculable, it is considered that a vector cannot be divided by another vector.

If vector $\boldsymbol{D}$ cannot be obtained from $\boldsymbol{C}$ through the relationship T between vectors $\boldsymbol{A}$ and vector $\boldsymbol{B}$, we can regard vector mathematics as incomplete. The tensor was formulated to resolve this issue. Next, we explain the solution by the tensor. In (15.2), B_x, B_y, and B_z are assumed as primary functions of A_x, A_y, and A_z, that is, they do not contain secondary A_x^2 or tertiary A_x^3. At this time, (15.2) is rewritten as follows:

$$\begin{aligned}B_x\boldsymbol{e_x} + B_y\boldsymbol{e_y} + B_z\boldsymbol{e_z} = (a_1A_x + a_2A_y + a_3A_z)\boldsymbol{e_x}\\ + (b_1A_x + b_2A_y + b_3A_z)\boldsymbol{e_y}\\ + (c_1A_x + c_2A_y + c_3A_z)\boldsymbol{e_z}.\end{aligned} \tag{15.4}$$

The calculations can be much simplified by compacting (15.4) in a matrix form. Using the matrix multiplication notation

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} a_1A_x + a_2A_y + a_3A_z \\ b_1A_x + b_2A_y + b_3A_z \\ c_1A_x + c_2A_y + c_3A_z \end{pmatrix},$$

we rewrite (15.4) as follows:

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}. \tag{15.5}$$

Comparing (15.5) and

$$\boldsymbol{B} = T\boldsymbol{A}, \tag{15.1}$$

we can write

$$T = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}, \tag{15.6}$$

where T is a tensor. Using T, we can construct vector $\boldsymbol{D}$ from vector $\boldsymbol{C}$. Since $\boldsymbol{D} = T\boldsymbol{C}$, we observe that

$$\begin{aligned} \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} &= T \begin{pmatrix} C_x \\ C_y \\ C_z \end{pmatrix} \\ &= \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} C_x \\ C_y \\ C_z \end{pmatrix} \\ &= \begin{pmatrix} a_1C_x + a_2C_y + a_3C_z \\ b_1C_x + b_2C_y + b_3C_z \\ c_1C_x + c_2C_y + c_3C_z \end{pmatrix}. \end{aligned}$$

The above result exemplifies how the tensor construct can express the relationship between two vectors that are indivisible in their vector form.

Once we appreciate that tensors express relationships between two vectors, we can understand that tensors cannot be drawn on a figure. If an equation exists that changes a triangle into a quadrangle, both the triangle and the quadrangle can be drawn as figures but the conversion formula cannot be represented in this manner. Similarly, we can draw two vectors on a figure, but not the tensor that relates them.

As mentioned above, the tensor construct overcomes the divisibility problem inherent in vector. However, we can also obtain vector $\boldsymbol{D}$ by division of new octonions. In the new octonion notation, three vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are expressed as

$$A = A_xi + A_yj + A_zk,\ B = B_xi + B_yj + B_zk,\ C = C_xi + C_yj + C_zk, \tag{15.7}$$

and the equation

$$\boldsymbol{B} = T\boldsymbol{A} \tag{15.1}$$

becomes

$$B = HA. \tag{15.8}$$

H is assumed as the function relating $\boldsymbol{A}$ and $\boldsymbol{B}$. As this function completely differs from the tensor, it is denoted as H. H implies the inclusion of the fourth imaginary number h.

Right-multiplying both sides of (15.8) by $\overline{A}$, we obtain

$$\begin{aligned} B\overline{A} &= HA\overline{A} \\ &= H\left|A\right|^2. \end{aligned}$$

Thus, if $A \neq 0$, we have

$$H = \frac{B\overline{A}}{|A|^2}.$$

Since $\boldsymbol{D}$ is the product of $\boldsymbol{C}$ and H, we write

$$\begin{aligned} D &= HC \\ &= \frac{B\overline{A}}{|A|^2}C. \end{aligned} \tag{15.9}$$

Equation (15.9) shows that vector division is possible if the vectors are rewritten as new octonions. Thus, $\boldsymbol{D}$ can be algebraically calculated without invoking tensors. Furthermore, this calculation is manageable by junior high school students, whereas tensor calculations are difficult even for college students.

As mentioned above, if a new octonion is used, a vector can be easily constructed from other vectors without using tensors. However, this discussion applies only to straight coordinates and is not realized in curved coordinate axes. This limitation occurs because the multiplication of the two new octonions in (15.8) is realized only in straight coordinates. Whether coordinate rotation by a new octonion is realized in curvilinear coordinates is yet to be proven. In relativity theory, special and general versions are formulated in straight and curvilinear coordinates, respectively. Here we restrict our discussion to special relativity.

15.2 Differences between the tensor and the new octonion

The previous section demonstrated how, similar to the tensor, a vector can be constructed from another vector by expressing the vectors as new octonions. In this section, we explain the differences between the tensor and the new octonion. To simplify the formulas, coordinate components are expressed in Hamilton's notation. That is, i, j, and k are not attached to their coordinate components.

Assume that four points $A(1,\ 1,\ 1)$, $B(-1,\ 1,\ 1)$, $C(-1,\ -1,\ -1)$, and $D(1,\ -1,\ -1)$ are distributed in three-dimensional space, as shown in Figure 15.1. Clearly, points A, B, C, and D exist in the same plane, leaning at 45° to the x-y plane. Denote the vectors connecting the origin O and points A, B, C, and D as $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$, respectively, and let vector $\boldsymbol{B}$ be related to vector $\boldsymbol{A}$, through the tensor T:

$$\boldsymbol{B} = T\boldsymbol{A}.$$

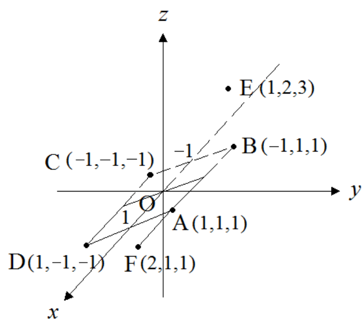


Figure 15.1

In a matrix notation, this equation becomes

$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{15.10}$$

We now obtain the tensor T that realizes (15.10). Substituting the tensor introduced in the previous section

$$T = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \tag{15.6}$$

into (15.10) yields

$$\begin{aligned} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + a_2 + a_3 \\ b_1 + b_2 + b_3 \\ c_1 + c_2 + c_3 \end{pmatrix}. \end{aligned}$$

This equation is equivalent to

$$\begin{aligned} -1 &= a_1 + a_2 + a_3, \\ 1 &= b_1 + b_2 + b_3, \\ 1 &= c_1 + c_2 + c_3. \end{aligned}$$

As we have three equations in nine unknowns, this system cannot be solved. Thus, the unknown tensor components are labeled a, b, and c, and all other elements are assumed as 0. For convenient arrangement of a, b, and c, we also introduce a tensor that is symmetric to the straight line drawn from the upper left to the lower right. This tensor is written as follows:

$$T = \begin{pmatrix} a\ b\ c \\ b\ 0\ 0 \\ c\ 0\ 0 \end{pmatrix}. \tag{15.11}$$

Substituting this tensor into (15.10), we obtain

$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a\ b\ c \\ b\ 0\ 0 \\ c\ 0\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} a+b+c \\ b \\ c \end{pmatrix}.$$

Thus, we have

$$a+b+c = -1,\ b = 1,\ c = 1,$$

from which

$$a = -3,\ b = c = 1.$$

Substituting these results into (15.11) yields

$$T = \begin{pmatrix} -3\ 1\ 1 \\ 1\ 0\ 0 \\ 1\ 0\ 0 \end{pmatrix}. \tag{15.12}$$

This tensor constructs vector $\boldsymbol{B}$ from vector $\boldsymbol{A}$.

We now use (15.12) to construct a vector from vector $\boldsymbol{C}$. Since the components of $\boldsymbol{C}$ are $(-1,\ -1,\ -1)$, we can write

$$T\boldsymbol{C} = \begin{pmatrix} -3\ 1\ 1 \\ 1\ 0\ 0 \\ 1\ 0\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} 3-1-1 \\ -1 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

This is vector $\boldsymbol{D}$. That is, the tensor T that constructs vector $\boldsymbol{B}$ from vector $\boldsymbol{A}$ also constructs vector $\boldsymbol{D}$ from vector $\boldsymbol{C}$.

Comparing $\triangle OAB$ described by $\boldsymbol{A}$ and $\boldsymbol{B}$ with $\triangle OCD$ described by $\boldsymbol{C}$ and $\boldsymbol{D}$, we observe from Figure 15.1 that the triangles are congruent. In other words, $\angle AOB$ equals $\angle COD$ and the lengths of the three sides are identical. If the coordinate components of point B are doubled to $(-2,\ 2,\ 2)$, the tensor (15.12) is also doubled. Specifically, T becomes

$$T = \begin{pmatrix} -6 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

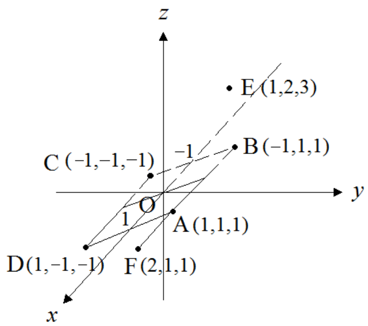


Figure 15.1

If vector $\boldsymbol{C}$ is multiplied by this tensor, the coordinate components of point D are also doubled to $(2,\ -2,\ -2)$. That is, $\triangle OCD$ and $\triangle OAB$ remain similar under size scaling operations. Moreover, when the size of vector $\boldsymbol{C}$ is $d \times \boldsymbol{A}$, $\triangle OCD$ is converted to a similar triangle of size $d \times \triangle OAB$. These results are easily proved by calculations.

From the above results, if vector $\boldsymbol{D}$ is the product of $\boldsymbol{C}$ and T, (as vector $\boldsymbol{B}$ is the product of $\boldsymbol{A}$ and T), we find that $\triangle OAB$ and $\triangle OCD$ are similar. This is a fundamental property of tensors. We now discuss the differences between the tensor and the new octonion.

(1) The first difference between the tensor and the new octonion

The first difference between the tensor and the new octonion is that the new octonion offers a unique solution. Equation (15.11) is one of multiple tensors T that construct a vector $\boldsymbol{B}$ from vector $\boldsymbol{A}$. Because (15.11) was temporarily assumed, the existence of other tensors was not considered in our previous example. As an example, consider the following tensors:

$$T = \begin{pmatrix} a\ 0\ b \\ 0\ 0\ c \\ b\ c\ 0 \end{pmatrix}. \tag{15.13}$$

Substituting this tensor into (15.10) yields

$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a\ 0\ b \\ 0\ 0\ c \\ b\ c\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} a+b \\ c \\ b+c \end{pmatrix},$$

from which we obtain

$$a+b=-1,\ c=1,\ b+c=1,$$

and therefore

$$a=-1,\ b=0,\ c=1.$$

Substituting these results into (15.13) gives

$$T = \begin{pmatrix} -1\ 0\ 0 \\ 0\ 0\ 1 \\ 0\ 1\ 0 \end{pmatrix}. \tag{15.14}$$

Multiplying vector $\boldsymbol{C}$ by this tensor, we find that

$$\begin{pmatrix} -1\ 0\ 0 \\ 0\ 0\ 1 \\ 0\ 1\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix},$$

which is again vector $\boldsymbol{D}$. Therefore, this tensor also indicates a similarity between $\triangle OAB$ and $\triangle OCD$.

Other tensors that relate vector $\boldsymbol{B}$ to $\boldsymbol{A}$ and vector $\boldsymbol{D}$ to $\boldsymbol{C}$ are

$$\begin{pmatrix} 0\ b\ c \\ b\ a\ 0 \\ c\ 0\ 0 \end{pmatrix}, \begin{pmatrix} 0\ b\ c \\ b\ 0\ 0 \\ c\ 0\ a \end{pmatrix}, \begin{pmatrix} 0\ 0\ b \\ 0\ a\ c \\ b\ c\ 0 \end{pmatrix}, \begin{pmatrix} 0\ 0\ b \\ 0\ 0\ c \\ b\ c\ a \end{pmatrix}, \begin{pmatrix} a\ 0\ 0 \\ 0\ b\ 0 \\ 0\ 0\ c \end{pmatrix}, \begin{pmatrix} 0\ a\ b \\ a\ 0\ c \\ b\ c\ 0 \end{pmatrix}.$$

That is, vector $\boldsymbol{B}$ can be constructed from vector $\boldsymbol{A}$ in multiple ways, all of which are mathematically valid.

We now show that a unique new octonion constructs vector $\boldsymbol{B}$ from vector $\boldsymbol{A}$. The new octonions A, B, C, and D describing vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$, respectively, are

$$A = i + j + k,\ B = -i + j + k,\ C = -i - j - k,\ D = i - j - k. \tag{15.15}$$

Denoting the new octonion that constructs vector $\boldsymbol{B}$ from vector $\boldsymbol{A}$ as H, we have

$$B = HA.$$

Substituting (15.15) into this expression, we obtain

$$-i + j + k = H(i + j + k).$$

Right-multiplying both sides of this equation by the new octonion conjugate $\overline{A} = -i - j - k$, the equations become

$$\begin{aligned}
(-i + j + k)(-i - j - k) &= H(i + j + k)(-i - j - k),\\
(-i + j + k)(i + j + k) &= H(i + j + k)(i + j + k),\\
-i^2 - ij - ik + ji + j^2 + jk + ki + kj + k^2 &= H(i^2 + ij + ik + ji + j^2 + jk + ki + kj + k^2),\\
1 - k + j - k - 1 + i + j - i - 1 &= H(-1 + k - j - k - 1 + i + j - i - 1),\\
-1 + 2j - 2k &= H(-3),\\
H &= (1 - 2j + 2k)/3. \qquad (15.16)
\end{aligned}$$

Equation (15.16) is the new octonion that constructs vector $\boldsymbol{B}$ from vector $\boldsymbol{A}$.

We now investigate whether vector $\boldsymbol{D}$ is obtained as the product of $\boldsymbol{C}$ and H. From (15.15) and (15.16), we obtain

$$\begin{aligned}
HC &= (1 - 2j + 2k)(-i - j - k)/3\\
&= (-i - j - k + 2ji + 2j^2 + 2jk - 2ki - 2kj - 2k^2)/3\\
&= (-i - j - k - 2k - 2 + 2i - 2j + 2i + 2)/3\\
&= (3i - 3j - 3k)/3\\
&= i - j - k,
\end{aligned}$$

which is precisely vector $\boldsymbol{D}$ from (15.15). That is, the new octonion

$$H = (1 - 2j + 2k)/3 \tag{15.16}$$

constructs vector $\boldsymbol{B}$ from vector $\boldsymbol{A}$ as well as vector $\boldsymbol{D}$ from vector $\boldsymbol{C}$. Moreover, H is a unique solution. This calculation clarifies the first difference between the tensor and the new octonion.

(2) The second difference between the tensor and the new octonion

We now explain the second difference between the tensor and the new octonion. In Figure 15.1, vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ lie on the same plane. Consider vector $\boldsymbol{E}$ extending from the origin O to a point $E(1,\ 2,\ 3)$ lying outside the plane.

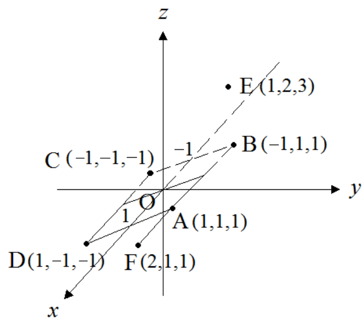


Figure 15.1

We examine the vector transformed from vector $\boldsymbol{E}$ by the tensor

$$T = \begin{pmatrix} -3\ 1\ 1 \\ 1\ 0\ 0 \\ 1\ 0\ 0 \end{pmatrix}, \tag{15.12}$$

which constructs vector $\boldsymbol{B}$ from vector $\boldsymbol{A}$. Explicit calculation gives the following equations:

$$\begin{aligned} T\boldsymbol{E} &= \begin{pmatrix} -3\ 1\ 1 \\ 1\ 0\ 0 \\ 1\ 0\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -3+2+3 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

That is, the tensor T transforms vector $\boldsymbol{E}$ into vector $\boldsymbol{F}$, which is the vector connecting the origin O to point $F(2,\ 1,\ 1)$. Moreover, as shown in Figure 15.1, point F lies on the plane occupied by points $A,\ B,\ C$, and D. Indeed, regardless of its location, if E is multiplied by the tensor T, the point F return to the plane occupied by points $A,\ B,\ C$, and D. This is another fundamental property of tensors.

We have established that $\triangle OAB$ and $\triangle OCD$ described by points $A,\ B,\ C$, and D on the same plane are similar. However, are $\triangle OAB$ and $\triangle OEF$ similar? By Pythagoras' theorem, the lengths of the sides OE and OF are calculated as

$$|OE|^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14,$$
$$|OF|^2 = 2^2 + 1^2 + 1^2 = 4 + 1 + 1 = 6.$$

As $|OE|^2 \neq |OF|^2$, $\triangle OAB$ and $\triangle OEF$ are not similar. Thus, while tensor T preserves congruency between $\triangle OAB$ and $\triangle OCD$ in the same plane, it does not construct congruent triangles from vectors lying outside the plane. In other words, the tensor is restricted to vectors lying in the same plane. Vectors lying outside the plane cannot be synchronized with vectors occupying a common plane. However, we emphasize that these discussions relate only to straight coordinates, i.e., the premises of special relativity, and may not apply to curvilinear coordinates, i.e., general relativity.

Unlike the tensor, vectors transformed by the new octonion can be synchronized throughout the entire three-dimensional space. The new octonion H that constructs vector $\boldsymbol{B}$ from vector $\boldsymbol{A}$ is uniquely specified as

$$H = (1 - 2j + 2k)/3. \tag{15.16}$$

Since the new octonion describing vector $\boldsymbol{E}$ is $E = i + 2j + 3k$, the new octonion of point G obtained as the product of E by H is

$$\begin{aligned} G &= HE \\ &= (1 - 2j + 2k)(i + 2j + 3k)/3 \\ &= (i + 2j + 3k - 2ji - 4j^2 - 6jk + 2ki + 4kj + 6k^2)/3 \\ &= (i + 2j + 3k + 2k + 4 - 6i + 2j - 4i - 6)/3 \\ &= (-2 - 9i + 4j + 5k)/3. \end{aligned} \tag{15.17}$$

Since G has four coordinate components, the vector $\boldsymbol{E}$ in three-dimensional space is transformed to a vector $\boldsymbol{G}$ in four-dimensional space–time.

For readers who cannot accept that a three-dimensional vector becomes a four-dimensional vector, we present a typical example. If a stationary observer A detects that a point mass D does not move in time (i.e., $t = 0$ always), the coordinates of D are denoted by the three-dimensional vector $(x,\ y,\ z)$. Substituting $t = 0$ into the Lorentz transformations

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{3.3}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \tag{3.4}$$

$$y' = y, \tag{3.5}$$

$$z' = z, \tag{3.6}$$

we have

$$t' = \frac{-(v/c^2)x}{\sqrt{1 - v^2/c^2}},$$

$$x' = \frac{x}{\sqrt{1 - v^2/c^2}},$$

$$y' = y,$$

$$z' = z.$$

The coordinates of D observed from a linearly moving observer B are described by a four-dimensional vector $(t',\ x',\ y',\ z')$. Thus, a three-dimensional vector naturally changes to a four-dimensional vector under a familiar coordinate transformation.

Algebraic theory dictates that possible worlds are restricted to one, two, four, or eight dimensions (concepts developed in this book permit overlapping four-dimensional worlds). Thus, independent three-dimensional space is forbidden by mathematics. In this context, a vector $\boldsymbol{E}$ in three-dimensional space naturally converts to a vector $\boldsymbol{G}$ in four-dimensional space–time.

Next, we investigate whether a semantic can be defined for a vector $\boldsymbol{G}$ in four-dimensional space–time constructed from vector $\boldsymbol{E}$ by

$$H = (1 - 2j + 2k)/3. \tag{15.16}$$

First, we find that

$$\begin{aligned}
|OE|^2 &= E\overline{E} \\
&= (i + 2j + 3k)(-i - 2j - 3k) \\
&= -i^2 - 2ij - 3ik - 2ji - 4j^2 - 6jk - 3ki - 6kj - 9k^2 \\
&= 1 - 2k + 3j + 2k + 4 - 6i - 3j + 6i + 9 \\
&= 14.
\end{aligned}$$

From

$$G = (-2 - 9i + 4j + 5k)/3, \tag{15.17}$$

we obtain

$$\begin{aligned}
|OG|^2 &= G\overline{G} \\
&= (-2 - 9i + 4j + 5k)(-2 + 9i - 4j - 5k)/9 \\
&= (4 - 18i + 8j + 10k + 18i - 81i^2 + 36ij + 45ik \\
&\quad - 8j + 36ji - 16j^2 - 20jk - 10k + 45ki - 20kj - 25k^2)/9 \\
&= (4 - 18i + 8j + 10k + 18i + 81 + 36k - 45j \\
&\quad - 8j - 36k + 16 - 20i - 10k + 45j + 20i + 25)/9 \\
&= (4 + 81 + 16 + 25)/9 \\
&= 126/9 \\
&= 14.
\end{aligned}$$

Thus, we have

$$|OE|^2 = |OG|^2 = 14. \tag{15.18}$$

These equations indicate that the four-dimensional space–time vector $\boldsymbol{G}$ is obtained by rotating the three-dimensional vector $\boldsymbol{E}$ in four-dimensional space–time.

Next, we calculate $|EG|^2$. If the new octonion connecting points E and G is denoted (EG), we have

$$\begin{aligned}
(EG) &= G - E \\
&= (-2 - 9i + 4j + 5k)/3 - (i + 2j + 3k) \\
&= -\frac{2}{3} - 3i + \frac{4}{3}j + \frac{5}{3}k - i - 2j - 3k \\
&= -\frac{2}{3} - 4i - \frac{2}{3}j - \frac{4}{3}k.
\end{aligned}$$

Thus, as

$$\begin{aligned}
|EG|^2 &= (EG)\overline{(EG)} \\
&= (-\frac{2}{3} - 4i - \frac{2}{3}j - \frac{4}{3}k)(-\frac{2}{3} + 4i + \frac{2}{3}j + \frac{4}{3}k) \\
&= (\frac{2}{3})^2 + 4^2 + (\frac{2}{3})^2 + (\frac{4}{3})^2 \\
&= \frac{4}{9} + 16 + \frac{4}{9} + \frac{16}{9} \\
&= \frac{24}{9} + 16
\end{aligned}$$

$$= \frac{8}{3} + 16$$
$$= \frac{56}{3},$$

we find that

$$|EG| = \sqrt{\frac{56}{3}} = \frac{2\sqrt{14}}{\sqrt{3}}. \tag{15.19}$$

From (15.18) and (15.19), the ratios of the sides of $\triangle OEG$ are

$$|OE| : \ |OG| : \ |EG| = \sqrt{14} : \ \sqrt{14} : \ \frac{2\sqrt{14}}{\sqrt{3}} = \sqrt{3} : \ \sqrt{3} : \ 2. \tag{15.20}$$

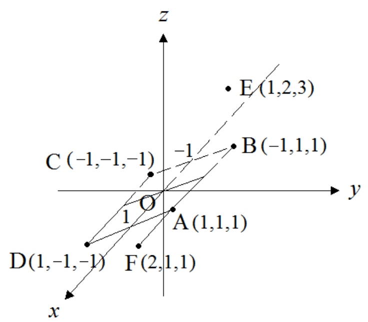


Figure 15.1

From Figure 15.1, the side lengths of $\triangle OAB$ are

$$\begin{aligned} |OA| &= |OB| \\ &= \sqrt{1^2 + 1^2 + 1^2} \\ &= \sqrt{3}, \\ |AB| &= 2. \end{aligned}$$

Thus, we find that

$$|OA| : \ |OB| : \ |AB| = \sqrt{3} : \ \sqrt{3} : \ 2. \tag{15.21}$$

Equations (15.20) and (15.21) demonstrate that $\triangle OEG$ and $\triangle OAB$ are similar.

Here the reader must recognize that $\triangle OEG$ exists in four-dimensional space–time, while $\triangle OAB$ is a three-dimensional spatial entity. That is, the new octonion transformation converts a vector in three-dimensional space into another vector in four-dimensional space–time, and realizes congruent triangles in four-dimensional space–time. As previously explained, the tensor is applicable only to vectors describing two-dimensional planes in three-dimensional space. The new octonion transformation is more general, being applicable to all vectors regardless of their relationships.

In three-dimensional linear coordinate spaces, the tensor is made to convert a three-dimensional vector to another three-dimensional vector. However, vectors occupying different planes cannot be synchronously calculated using tensor mathematics. In other words, when the tensor operates on a vector in three-dimensional space, it converts the vector to a two-dimensional vector. This is clear from Figure 15.1. As demonstrated previously, if any point E extending beyond the plane occupied by points A, B, C, and D is multiplied by a tensor T, the resulting point F returns to the plane occupied by points A, B, C, and D. As there are three coordinate components, we tend to consider that tensors manipulate three-dimensional vectors. However, correctly speaking, since tensor mathematics is applicable only to vectors in the same plane, the tensor confines the vectors to two dimensions and prevents the four-dimensional transformation of three-dimensional vectors.

Engineers would prefer that tensor transformation do not convert three-dimensional vectors to four-dimensional vectors. However, tensor mathematics may lead researchers of space–time or fundamental particles to incorrect conclusions. Considering a tensor of three columns and three rows and a tensor of four columns and four rows, the latter will convert a four-dimensional vector into a three-dimensional vector, which exists in a cross-section of four-dimensional space–time. Regardless of whether we intend to work with four-dimensional vectors, the result of tensor mathematics is a three-dimensional vector. This is easily demonstrated by performing transformations by a tensor of four columns and four rows. Moreover, a tensor of three columns and three rows does not elucidate the relationship between two-dimensional planes, while a tensor of four columns and four rows precludes us from understanding the relationships between cross-sections in four-dimensional space–time. If we use tensors to examine the structure of four-dimensional space–time, we will obtain a sum of many cross-sections rather than appreciating the true nature of space–time.

15.3 Kronecker δ

The Kronecker δ (delta), which commonly appears in tensor and relativity texts, is defined as

$$\delta^i_j = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} .$$

Some readers may regard δ to be somewhat incongruous. The reason for this discomfort is that variables, whose values change according to conditions, are typically absent in mathematically elegant geometries and algebras. Given a function $y = x$, if $x > 0$, then $y > 0$, and if $x < 0$, then $y < 0$. This causes no concern, since y is a function. In pure geometry and algebra, the parameters are fixed, unlike δ, which is conditional.

In three-dimensional space, i and j denote the natural numbers $1-3$. The matrix form of δ^i_j is

$$\delta^i_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiplying this matrix by a vector $\boldsymbol{A}(1,\ 1,\ 1)$, we obtain

$$\begin{aligned} \delta^i_j \boldsymbol{A} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \boldsymbol{A}. \end{aligned}$$

Since the δ^i_j transformation retrieves the vector, it is a tensor that constructs the same vector.

Next, we calculate δ^i_j in terms of the new octonion. We denote the new octonion that changes vector $\boldsymbol{A}(1,\ 1,\ 1)$ into vector $\boldsymbol{A}$ is assumed as H. Since

$$A = HA,$$

we can write

$$i + j + k = H(i + j + k).$$

Right-multiplying both sides of this equation by the new octonion conjugate $\overline{A} = -i - j - k$ of $A = i + j + k$, the equations become

$$\begin{aligned} (i + j + k)(-i - j - k) &= H(i + j + k)(-i - j - k), \\ 1 + 1 + 1 &= H(1 + 1 + 1), \\ 3 &= H3, \\ H &= 1. \end{aligned}$$

That is, Kronecker δ^i_j is 1 by the new octonion and the number does not conditionally change its value.

As explained in Section 15.2, the tensor collapses the three-dimensional vector into a two-dimensional plane and prevents the construction of a four-dimensional vector. Therefore, tensor mathematics requires the Kronecker δ, which conditionally changes its value, and is thus a cause for concern among some purists. Since Kronecker δ is unconditionally 1 in the new octonion interpretation, it is not required in new octonion space–time investigations. However, since our discussion is restricted to straight coordinates, we may draw different conclusions in curvilinear coordinates.

15.4 Metric tensor

An important tensor in Einstein's relativity theory is the metric (distance) tensor, also known as the fundamental tensor. This section relates the new octonion to the metric tensor.

General relativity theory generalizes the time ct and the distances $x,\ y,\ z$, which are written as $X^0,\ X^1,\ X^2$, and X^3 in special relativity. Thus, the square of the world distance s

$$s^2 = (ct)^2 - x^2 - y^2 - z^2 \tag{7.1}$$

is written in relativity textbooks as

$$s^2 = (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2. \tag{15.22}$$

However, as explained in Section 7.2, the world distance in terms of the new octonion is

$$s^2 = -(ct)^2 + x^2 + y^2 + z^2. \tag{7.6}$$

Note that Equation (7.1) formulates the world distance in the negative world.

Equation (15.22) is compactly rewritten in terms of η (eta), μ (mu), and ν (nu) as

$$s^2 = \eta_{\mu\nu} X^\mu X^\nu. \tag{15.23}$$

However,

$$\eta_{00} = 1,\ \eta_{11} = \eta_{22} = \eta_{33} = -1,\ \eta_{\mu\nu} = 0.\ (\because\ \mu \neq \nu) \tag{15.24}$$

In tensor form, (15.24) is written as

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{15.25}$$

Equation (15.25) defines a metric tensor in special relativity. Similar to (15.23), we can rewrite the four-dimensional infinitesimal world distance ds as

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu, \tag{15.26}$$

where ds^2 is an alternative form of $(ds)^2$.

What is the result of applying the new octonion to the above calculation? In the notation X^0, X^1, X^2, and X^3, the new octonion describing a world point is

$$A = X^0 h + X^1 i + X^2 j + X^3 k.$$

The square of the world distance s becomes

$$\begin{aligned} s^2 &= A\overline{A} \\ &= (X^0 h + X^1 i + X^2 j + X^3 k)(X^0 h - X^1 i - X^2 j - X^3 k) \\ &= -(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2. \end{aligned} \tag{15.27}$$

As explained in Section 7.2, (15.27) is the mathematically calculated world distance. Similarly, the square of the infinitesimal four-dimensional world distance ds becomes

$$\begin{aligned} ds^2 &= dA\overline{dA} \\ &= (dX^0 h + dX^1 i + dX^2 j + X^3 k)(dX^0 h - dX^1 i - dX^2 j - dX^3 k) \\ &= -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2. \end{aligned}$$

The square of the world distance s can be calculated from the new octonion A and its conjugate $\overline{A}$ avoiding the need for metric tensors such as (15.23) and (15.26). As explained in Section 15.2, tensor mathematics calculates physical quantities in four-dimensional space–time by confining them to a cross section of four-dimensional space–time. In contrast, the new octonion mathematics calculates physical quantities throughout the entire four-dimensional space–time. Thus, tensors are not required in straight coordinate systems. However, this situation may alter in curvilinear coordinate systems.

16

Synchrotron Radiation

16.1 About synchrotron radiation

Consider an electron moving with uniform horizontal velocity v watched by a stationary observer. The situation is illustrated in Figure 16.1 (a) and (b). The coordinate axes of the static system are x, y, and z, while those of the moving electron are x', y', and z'. Moreover, the x- and x'-axes overlap. The y- and y'-axes as well as the z- and z'-axes point in the positive direction. When the electron moving at near-light velocity emits light in the y'-axial direction, how does the light appear to the stationary observer? The light seen by this observer is called synchrotron radiation.

Because relativity theory and the new octonion space–time theory yield different Lorentz transformations of y' and z', the two approaches may lead to different conclusions. This chapter focuses on the conclusions obtained from both approaches, and discusses their differences.

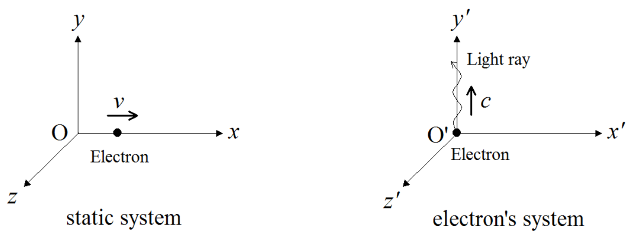


Figure 16.1 (a) Figure 16.1 (b)

16.2 Proof under Lorentz transformations

First, we use the Lorentz transformation of special relativity to explain the direction of synchrotron radiation. This content is written in standard relativity texts. The Lorentz-transformed velocity was discussed in Section 10.4.

Let the velocity of a point mass D be $V(V_x,\ V_y,\ V_z)$ with respect to a stationary observer A, and $V'(V'_x,\ V'_y,\ V'_z)$ with respect to an observer B moving along a straight line with uniform velocity v in the x-direction of A. Then, we have

$$V'_x = \frac{V_x - v}{1-(v/c^2)V_x}, \tag{10.16}$$

$$V'_y = \frac{V_y\sqrt{1-v^2/c^2}}{1-(v/c^2)V_x}, \tag{10.17}$$

$$V'_z = \frac{V_z\sqrt{1-v^2/c^2}}{1-(v/c^2)V_x}. \tag{10.18}$$

Now, consider that D is the light emitted in the y'-direction by an electron with velocity components $V'_x = 0,\ V'_y = c$, and $V'_z = 0$. Inserting $V'_x = 0$ into (10.16), we find that

$$0 = \frac{V_x - v}{1-(v/c^2)V_x},$$

$$V_x = v, \tag{16.1}$$

while inserting $V'_y = c$ into (10.17) yields

$$c = \frac{V_y\sqrt{1-v^2/c^2}}{1-(v/c^2)V_x}.$$

Substituting (16.1) into the above equation, we obtain

$$\begin{aligned} c &= \frac{V_y\sqrt{1-v^2/c^2}}{1-(v/c^2)v} \\ &= \frac{V_y\sqrt{1-v^2/c^2}}{1-v^2/c^2} \\ &= \frac{V_y}{\sqrt{1-v^2/c^2}}, \end{aligned}$$

$$V_y = c\sqrt{1-v^2/c^2}. \tag{16.2}$$

In addition, inserting $V'_z = 0$ into (10.18) yields

$$0 = \frac{V_z\sqrt{1-v^2/c^2}}{1-(v/c^2)V_x},$$

$$V_z = 0. \tag{16.3}$$

Collectively, we have

$$V_x = v, \tag{16.1}$$

$$V_y = c\sqrt{1-v^2/c^2}, \tag{16.2}$$

$$V_z = 0. \tag{16.3}$$

As the velocity of the electron v approaches the velocity of light c, (16.1), (16.2), and (16.3) become

$$V_x = c,$$

$$V_y = 0,$$

$$V_z = 0.$$

These equations indicate that, according to the stationary observer, light moves in the forward x-direction with velocity c. That is, in special relativity, light, which is emitted in the perpendicular direction by an electron moving with velocity v approaching that of light, is measured as it concentrates ahead of the stationary observer. The situation is shown in Figure 16.2.

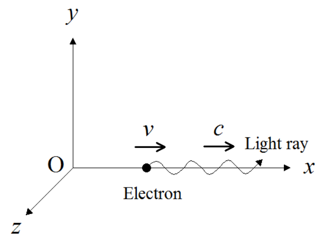


Figure 16.2

16.3 Proof under the new Lorentz transformations

Next, we derive the direction of the synchrotron radiation from the new Lorentz transformation. The velocities under the new Lorentz transformation are given by (for proof, see Section 10.5)

$$V'_x = \frac{V_x - v}{1 - (v/c^2)V_x}, \tag{10.25}$$

$$V'_y = \frac{V_y + (v/c)V_z h}{1 - (v/c^2)V_x}, \tag{10.26}$$

$$V'_z = \frac{V_z - (v/c)V_y h}{1 - (v/c^2)V_x}. \tag{10.27}$$

When D is the light emitted in the y'-direction by an electron, its velocity components are $V'_x = 0$, $V'_y = c$, and $V'_z = 0$. Inserting $V'_x = 0$ into (10.25), we obtain

$$0 = \frac{V_x - v}{1 - (v/c^2)V_x},$$

$$V_x = v. \tag{16.4}$$

Inserting $V'_y = c$ into (10.26) yields

$$c = \frac{V_y + (v/c)V_z h}{1 - (v/c^2)V_x}.$$

Substituting (16.4) into the above equation, we get

$$c = \frac{V_y + (v/c)V_z h}{1 - v^2/c^2}. \tag{16.5}$$

In addition, inserting $V'_z = 0$ into (10.27) yields

$$0 = \frac{V_z - (v/c)V_y h}{1 - (v/c^2)V_x},$$

$$V_z = (v/c)V_y h. \tag{16.6}$$

Substituting (16.6) into (16.5), we obtain

$$c = \frac{V_y(1 - v^2/c^2)}{1 - v^2/c^2},$$

$$V_y = c. \tag{16.7}$$

Substituting (16.7) into (16.6) gives

$$V_z = vh. \tag{16.8}$$

Collectively, (16.4), (16.7), and (16.8) are given by

$$V_x = v, \tag{16.4}$$

$$V_y = c, \tag{16.7}$$

$$V_z = vh. \tag{16.8}$$

Comparing these equations to those obtained by the Lorentz transformation

$$V_x = v, \tag{16.1}$$

$$V_y = c\sqrt{1 - v^2/c^2}, \tag{16.2}$$

$$V_z = 0, \tag{16.3}$$

we see that in the new Lorentz transformation, V_y is always c and V_z is a [zhk]-axis component vh in the negative world.

16.4 Synchrotron radiation and the constancy of the velocity of light

We obtained (16.1), (16.2), and (16.3) from the Lorentz transformation and (16.4), (16.7), and (16.8) from the new Lorentz transformation. Which is correct? First, we investigate whether (16.1), (16.2), and (16.3) are consistent with a constant velocity of light c. The square of the synthetic velocity V in terms of its components V_x, V_y, and V_z is

$$V^2 = V_x^2 + V_y^2 + V_z^2.$$

Substituting

$$V_x = v, \tag{16.1}$$

$$V_y = c\sqrt{1 - v^2/c^2}, \tag{16.2}$$

$$V_z = 0 \tag{16.3}$$

into this equation, we have

$$\begin{aligned} V^2 &= v^2 + c^2(1 - v^2/c^2) \\ &= c^2, \\ V &= c. \end{aligned}$$

This result fulfills the condition that c is constant.

Repeating the calculations with

$$V_x = v, \tag{16.4}$$

$$V_y = c, \tag{16.7}$$

$$V_z = vh \tag{16.8}$$

obtained by the new Lorentz transformation, we find that

$$\begin{aligned} V^2 &= v^2 + c^2 - v^2 \\ &= c^2. \end{aligned}$$

Thus, (16.4), (16.7), and (16.8) also fulfill the condition that c is constant.

16.5 Slant of synchrotron radiation

We now investigate the slant of light emitted in the y'-direction by an electron, as observed in the static system. By the Lorentz transformation, we have

$$V_x = v, \tag{16.1}$$

$$V_y = c\sqrt{1 - v^2/c^2}, \tag{16.2}$$

$$V_z = 0. \tag{16.3}$$

Thus, the slant of light with respect to the x-axis, defined as V_y/V_x, becomes

$$\frac{V_y}{V_x} = \frac{c\sqrt{1 - v^2/c^2}}{v}.$$

As the electronic velocity v approaches the velocity of light c, the limiting slant becomes

$$\begin{aligned}\lim_{v\to c} \frac{V_y}{V_x} &= \lim_{v\to c} \frac{c\sqrt{1 - v^2/c^2}}{v} \\ &= \frac{c\sqrt{1 - c^2/c^2}}{c} \\ &= 0.\end{aligned}$$

That is, the light travels along the x-axis in a straight line of gradient 0. For this reason, synchrotron radiation concentrates ahead of the stationary observer A.

If the calculations are repeated using the formulae (16.4), (16.7), and (16.8) obtained from the new Lorentz transformation, we require a different approach. In applying the Lorentz transformation to compute the slant, we only need to consider the x-y plane, since $V_z = 0$. However, in the new Lorentz transformation, the calculations are complicated by the $V_z = vh$ term.

To advance discussions, we propose a new axiom. In four-dimensional space–time with a double structure, if the velocity has three components in the x-, y-, and z-directions, we must first calculate the combined velocity V_{yz} in the y-z plane, and then calculate V_{yz}/V_x to obtain the gradient with respect to the x-axis. This calculation is correct in the positive world. However, it has not been proven correct when the velocity has a component in the negative world, such as $V_z = vh$. Thus, we declare this situation as an axiom.

First, we calculate the gradient with respect to the x-axis. From the equations

$$V_y = c, \tag{16.7}$$

$$V_z = vh, \tag{16.8}$$

we find that

$$\begin{aligned} V_{yz}^2 &= V_y^2 + V_z^2 \\ &= c^2 + (vh)^2 \\ &= c^2 - v^2. \end{aligned}$$

Thus, conditional on $c > v$, we have

$$V_{yz} = \sqrt{c^2 - v^2}. \tag{16.9}$$

From (16.9) and the x velocity component obtained from the new Lorentz transformation, i.e.,

$$V_x = v, \tag{16.4}$$

the gradient with respect to the x-axis of the synchrotron radiation is

$$\frac{V_{yz}}{V_x} = \frac{\sqrt{c^2 - v^2}}{v}.$$

As the electron velocity v approaches the velocity of light c, the limiting slant is obtained as

$$\begin{aligned} \lim_{v \to c} \frac{V_{yz}}{V_x} &= \lim_{v \to c} \frac{\sqrt{c^2 - v^2}}{v} \\ &= \frac{\sqrt{c^2 - c^2}}{c} \\ &= 0. \end{aligned} \tag{16.10}$$

That is, the light travels along the x-axis in a straight line of gradient 0, consistent with the result obtained by the Lorentz transformation.

Next, we calculate the gradient with respect to the y-axis. From the x and z velocity components obtained under the new Lorentz transformation, i.e.,

$$V_x = v, \tag{16.4}$$

$$V_z = vh, \tag{16.8}$$

we find that

$$\begin{aligned} V_{xz}^2 &= V_x^2 + V_z^2 \\ &= v^2 - v^2 \\ &= 0, \\ V_{xz} &= 0. \end{aligned} \tag{16.11}$$

From (16.11) and the y velocity component obtained under the new Lorentz transformation, i.e.,

$$V_y = c, \tag{16.7}$$

we have

$$\begin{aligned} \frac{V_{xz}}{V_y} &= \frac{0}{c} \\ &= 0. \end{aligned} \tag{16.12}$$

This result indicates that synchrotron radiation is observed as light in the y-direction, regardless of the electron velocity v.

According to (16.10), which obtains the gradient with respect to the x-axis, the synchrotron radiation must be directed along the x-axis. However, in (16.12), it also becomes oriented in the y-direction. What is the cause of this contradiction?

Before pursuing this result, we calculate the gradient with respect to the z-axis. From the x and y velocity components under the new Lorentz transformation, i.e.,

$$V_x = v, \tag{16.4}$$

$$V_y = c, \tag{16.7}$$

we obtain

$$\begin{aligned} V_{xy}^2 &= V_x^2 + V_y^2 \\ &= v^2 + c^2, \\ V_{xy} &= \sqrt{v^2 + c^2}. \end{aligned} \tag{16.13}$$

From (6.13) and the z velocity component obtained under the new Lorentz transformation, i.e.,

$$V_z = vh, \tag{16.8}$$

we find that

$$\begin{aligned} \frac{V_{xy}}{V_z} &= \frac{\sqrt{v^2 + c^2}}{vh} \\ &= -h\sqrt{1 + c^2/v^2}. \end{aligned}$$

As the electron velocity v approaches the velocity of light c, the limiting slant becomes

$$\begin{aligned} \lim_{v \to c} \frac{V_{xy}}{V_z} &= -\lim_{v \to c} h\sqrt{1 + c^2/v^2} \\ &= -h\sqrt{1 + c^2/c^2} \\ &= -\sqrt{2}h. \end{aligned}$$

According to this result, the gradient with respect to the z-axis extends into the negative world, and is hence invisible to us.

From the above results, the light emitted in the y'-direction from an electron moving in the x-direction at near-light velocity is observed by a stationary observer as light traveling in both the x- and y-directions. In addition, a component of the light exists in the negative world, but cannot be observed.

This result leads to a contradiction that the light emitted as a single world line from the moving electron is observed as the three world lines by a stationary observer. However, if we add the temporal component cth to the velocity components obtained under the new Lorentz transformation, i.e.,

$$V_x = v, \tag{16.4}$$

$$V_y = c, \tag{16.7}$$

$$V_z = vh, \tag{16.8}$$

we obtain the new octonion of the world line of light:

$$cth + V_x ti + V_y tj + V_z tk = cth + vti + ctj + vthk.$$

This equation implies a single world line observed in the static system; hence, the appearance of the three world lines is an artifact. The reasons for this anomaly are presently unknown. However, since the hk component resides in the negative world, the world line of the light emitted in the y'-direction ranges over both positive and negative worlds.

The axiom proposed above, that is, the gradient with respect to the x-axis is calculable if we first calculate V_{yz}/V_x, may be incorrect in four-dimensional space–time with double structure. Alternatively, the new octonion itself may be incorrect. However, in our analysis of special relativity, the direction of synchrotron radiation alone is inconsistent with the new octonion results. A correct interpretation of this result may be realized in future.

16.6 Observation of the perpendicular light of synchrotron radiation

In the previous section, we mentioned that the light emitted in the y'-direction from an electron moving in the x-direction at near-light velocity apparently arrives at a stationary observer from both the x- and y-directions. This raises the question: how is light traveling at velocity c in the y-direction perceived by the observer?

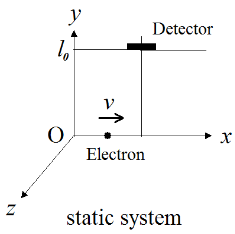


Figure 16.3 (a)

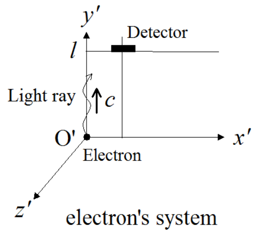


Figure 16.3 (b)

Consider a light detector placed on a straight line parallel to the direction of a moving electron, as shown in Figure 16.3 (a) and (b). The distance between the parallel lines is l_0 in the static system and l in the reference frame of the electron. We investigate the relationship between l_0 and l. Inserting $z' = 0$, $y' = l$, and $y = l_0$ into the new Lorentz transformations

$$y' = \frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}}, \tag{10.5}$$

$$z' = \frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}}, \tag{10.6}$$

we have

$$l = \frac{l_0 + (v/c)zh}{\sqrt{1 - v^2/c^2}}, \tag{16.14}$$

$$0 = \frac{z - (v/c)l_0 h}{\sqrt{1 - v^2/c^2}}. \tag{16.15}$$

From (16.15), we have

$$z = (v/c)l_0 h. \tag{16.16}$$

Substituting this result into (16.14), we obtain

$$\begin{aligned} l &= \frac{l_0 - (v/c)^2 l_0}{\sqrt{1 - v^2/c^2}} \\ &= l_0\sqrt{1 - v^2/c^2}. \end{aligned}$$

According to this formula, if the velocity of the electron v approachs the velocity of light c, l reduces to 0 in the reference frame of the electron, and light reaches the detector within a very short time.

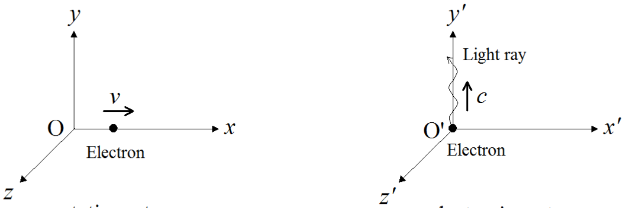
Conversely, in the reference frame of the stationary observer A, the light velocity in the y-direction is c and the distance to the detector is l_0. Thus, light reaches the detector in the time given by

$$\frac{l_0}{c}.$$

However, as shown in (16.16), the light velocity has a z-axial component in the negative world, whose detectability is unknown.

16.7 Light and inertial law

Does the locus of the light emitted in the y'-direction from the electron become a straight line in the reference frame of the electron in Figure 16.1 (a) and (b)?



static system

Figure 16.1 (a)

electron's system

Figure 16.1 (b)

Since light has dual wave–particle characteristics, it is called a photon in a particle context. In Figure 16.1, we ask whether the initial and final emitted photons can be traced from the same y'-axis. If the photon has mass, it will travel on a vertical line in the electron's reference frame. The situation is analogous to a ball thrown upwards in a moving train, which returns to the hand that supplied the motion. Since the law of inertia acts on particles with mass and the speed of the ball in the direction of the moving train matches that of the train, the ball returns to its original location. Similarly, if a photon has mass and travels with velocity v in the x-direction, the light (as seen by the electron) should trace a vertical line. However, according to special relativity, light possesses a kinetic mass, but no rest mass. Here, we investigate the locus of the light emitted from an electron without imposing the law of inertia.

As explained in Section 8.2, the world distance of light is always 0. Thus, if the world points of light in the static and moving systems are expressed as $(cth,\ xi,\ yj,\ zk)$ and $(ct'h,\ x'i,\ y'j,\ z'k)$, respectively, we can write

$$-(ct')^2 + x'^2 + y'^2 + z'^2 = -(ct)^2 + x^2 + y^2 + z^2 = 0. \tag{16.17}$$

If the photons emitted in the y'-direction from an electron follow a straight line, a world point on the locus is expressed as $A(ct'h,\ 0,\ ct'j,\ 0)$. This point fulfills the condition of the world line of light if its locus satisfies (16.17). Substituting A into the left-hand side of (16.17), we have

$$\begin{aligned}-(ct')^2 + x'^2 + y'^2 + z'^2 &= -(ct')^2 + 0 + (ct')^2 + 0 \\ &= 0.\end{aligned}$$

Thus, it is proved that the light emitted in the y' direction from an electron follows a straight line in the electron's reference frame

Consider a point B on the locus of light observed in the static system. In terms of the velocity components obtained from the new Lorentz transformation, i.e.,

$$V_x = v, \tag{16.4}$$

$$V_y = c, \tag{16.7}$$

$$V_z = vh, \tag{16.8}$$

B is expressed as $B(cth,\ vti,\ ctj,\ vthk)$. Substituting these coordinate components into the right-hand side of (16.17), we obtain

$$\begin{aligned}-(ct)^2 + x^2 + y^2 + z^2 &= -(ct)^2 + (vt)^2 + (ct)^2 + (vth)^2 \\ &= -(ct)^2 + (vt)^2 + (ct)^2 - (vt)^2 \\ &= 0.\end{aligned}$$

Therefore, the world line traced by B fulfills the condition of the world line of light. This result shows that the new Lorentz transformation correctly interprets light emitted perpendicularly from an electron.

Although light is massless in the above verification, we have realized the same phenomenon as the law of inertia. Some readers may argue that the law of inertia applies only to objects with mass. However, as explained in Section 9.3, the law of inertia can be paraphrased as follows: if no force is applied, the world line of a point mass is linear. In this context, mass is not required. In fact, Newton's law of inertia is precluded for massless entities, and should also apply to the kinetic mass of light. However, the kinetic mass is not required in the above verification.

16.8 Upthrow of a ball

In Section 16.5, we mentioned that the light emitted in the y'-direction from an electron moving in the x-direction at near-light velocity is observed in the stationary reference frame as lights emitted to both x- and y-directions. If a ball is perpendicularly thrown with velocity w in a train moving at velocity v, does the ball appear as two entities to a stationary observer, as in the case of light?

The upthrow of a ball in a moving train is expressed by inserting $V'_x = 0,\ V'_y = w$, and $V'_z = 0$ into the equations used in Section 16.3, i.e.,

$$V'_x = \frac{V_x - v}{1 - (v/c^2)V_x}, \tag{10.25}$$

$$V'_y = \frac{V_y + (v/c)V_z h}{1 - (v/c^2)V_x}, \tag{10.26}$$

$$V'_z = \frac{V_z - (v/c)V_y h}{1 - (v/c^2)V_x}. \tag{10.27}$$

Substituting $V'_x = 0$ into (10.25), we have

$$V_x = v, \tag{16.18}$$

while substituting $V'_z = 0$ into (10.27) yields

$$V_z = (v/c)V_y h. \tag{16.19}$$

From (10.26), (16.18), (16.19), and $V'_y = w$, we obtain

$$w = \frac{V_y - (v/c)^2 V_y}{1 - (v/c^2)v}.$$

Thus,

$$V_y = w. \tag{16.20}$$

In addition, from (16.19) and (16.20), we can write

$$V_z = (v/c)wh. \tag{16.21}$$

Collectively, the equations governing the upthrow of a ball in a moving train are

$$V_x = v, \tag{16.18}$$

$$V_y = w, \tag{16.20}$$

$$V_z = (v/c)wh. \tag{16.21}$$

Using these formulae, we can investigate the gradient with respect to the coordinate axis of the ball locus, as seen by the static observer. From (16.18) and (16.20), we have

$$\begin{aligned} V_{xy}^2 &= V_x^2 + V_y^2 \\ &= v^2 + w^2, \\ V_{xy} &= \sqrt{v^2 + w^2}. \end{aligned} \tag{16.22}$$

From (16.20) and (16.21), we also have

$$\begin{aligned} V_{yz}^2 &= V_y^2 + V_z^2 \\ &= w^2 - (v/c)^2 w^2, \\ V_{yz} &= w\sqrt{1 - v^2/c^2}. \end{aligned} \tag{16.23}$$

From (16.18) and (16.20), we obtain

$$\begin{aligned} V_{zx}^2 &= V_x^2 + V_z^2 \\ &= v^2 - (v/c)^2 w^2, \\ V_{zx} &= v\sqrt{1 - (w/c)^2}. \end{aligned} \tag{16.24}$$

The gradients with respect to the x-, y-, z-axes are obtained from (16.18), (16.20), (16.21), (16.22), (16.23), and (16.24) as follows:

$$\frac{V_{yz}}{V_x} = \frac{w\sqrt{1 - v^2/c^2}}{v} = w\sqrt{1/v^2 - 1/c^2}, \tag{16.25}$$

$$\frac{V_{zx}}{V_y} = \frac{v\sqrt{1 - (w/c)^2}}{w} = v\sqrt{1/w^2 - 1/c^2}, \tag{16.26}$$

$$\frac{V_{xy}}{V_z} = \frac{\sqrt{v^2 + w^2}}{(v/c)wh} = -ch\sqrt{1/v^2 + 1/w^2}. \tag{16.27}$$

If the ball is to appear as a single object in the stationary reference frame, (16.25) and (16.26) must multiply to unity. In other words, (16.25) and (16.26) must be the reciprocals of each other. Since this is not the case, a ball tossed upward in a moving train must be visible as two pieces from the static reference frame, besides the (non-observable) component that appears in the negative world.

The ball appears in one piece because v and w are very much less than the velocity of light c. When $1/c^2 = 0$, (16.25), (16.26), and (16.27) reduce to

$$\begin{aligned} \frac{V_{yz}}{V_x} &= w\sqrt{1/v^2} = w/v, \\ \frac{V_{zx}}{V_y} &= v\sqrt{1/w^2} = v/w, \\ \frac{V_{xy}}{V_z} &= -ch\sqrt{1/v^2 + 1/w^2} = -\infty. \end{aligned}$$

Since,

$$(V_{yz}/V_x)(V_{zx}/V_y) = (w/v)(v/w) = 1,$$

the ball converges to the same location along the x- and y-axes and is visible as one object from a static perspective. The reciprocal of $V_{xy}/V_z = -\infty$ is $V_z/V_{xy} = 0$, implying that the ball resides on the x-y plane, and no contradiction occurs.

It is hard to believe that a ball thrown upwards at velocity w in a train moving at velocity v moves perpendicularly in three directions, when observed by a stationary observer. This idea could be experimentally tested by a sufficiently precise detector.

17

Dynamics by the New Octonion

17.1 Formulae of the physical quantity that is constant by coordinate transformation

In Section 3.5, we obtained the formulae of the new Lorentz transformation of time and distance using the coordinate transformation $\overline{B}/|B|$ that makes the absolute value of world distance constant. Since we consider velocity, momentum, and acceleration in this chapter, we first obtain the conditions that should be fulfilled by physical quantities whose absolute values become constant, even if they are changed into the coordinates of the observer with linear uniform motion.

It is assumed that physical quantity P seen by stationary observer A has four components $(Eh,\ Fi,\ Gj,\ Hk)$ in four-dimensional space–time . E is a temporal component and F, G, and H are space components. Suppose that the coordinate components of the physical quantity P are $P'(E'h,\ F'i,\ G'j,\ H'k)$, as seen from B moving in the x-direction of A with uniform velocity v. The formulae of the coordinate transformation that makes the absolute value of P constant, i.e., $|P| = |P'|$, can be obtained by multiplying P by $\overline{B}/|B|$ when we assume the new octonion of observer B to be B, as explained in Section 3.5. Since $B = cth + vti$, we can write

$$P' = \frac{P(cth - vti)}{\sqrt{(cth + vti)(cth - vti)}}.$$

If $c > v$, substituting $P = Eh + Fi + Gj + Hk$ into this equation, we have

$$\begin{aligned} P' &= \frac{(Eh + Fi + Gj + Hk)(cth - vti)}{\sqrt{(cth + vti)(cth - vti)}} \\ &= \frac{1}{cth\sqrt{1 - (vti)^2/(cth)^2}} \\ &\quad \times \big(Ecth^2 + Fcthi + Gcthj + Hcthk - Evthi - Fvti^2 - Gvtji - Hvtki\big) \end{aligned}$$

$$= \frac{1}{cth\sqrt{1 - v^2/c^2}}$$
$$\times (-Ect + Fcthi + Gcthj + Hcthk - Evthi + Fvt + Gvtk - Hvtj)$$
$$= \frac{-h}{ct\sqrt{1 - v^2/c^2}}$$
$$\times [(-Ect + Fvt) + (Fct - Evt)hi + (Gcth - Hvt)j + (Hcth + Gvt)k]$$
$$= \frac{1}{\sqrt{1 - v^2/c^2}}$$
$$\times [(E - Fv/c)h + (F - Ev/c)i + (G + Hvh/c)j + (H - Gvh/c)k] .$$

Since $P' = E'h + F'i + G'j + H'k$, by comparing the coefficients, we find that

$$E' = \frac{E - (v/c)F}{\sqrt{1 - v^2/c^2}}, \tag{17.1}$$

$$F' = \frac{F - (v/c)E}{\sqrt{1 - v^2/c^2}}, \tag{17.2}$$

$$G' = \frac{G + (v/c)Hh}{\sqrt{1 - v^2/c^2}}, \tag{17.3}$$

$$H' = \frac{H - (v/c)Gh}{\sqrt{1 - v^2/c^2}}, \tag{17.4}$$

where (17.1), (17.2), (17.3), and (17.4) are formulae in which the physical quantity that does not change in absolute value should fulfill, even if it is seen by observer B undergoing linear uniform motion.

We can confirm $|P| = |P'|$ by substituting (17.1), (17.2), (17.3), and (17.4) into

$$|P'|^2 = (E'h + F'i + G'j + H'k)(E'h - F'i - G'j - H'k),$$

and having

$$|P'|^2 = -E'^2 + F'^2 + G'^2 + H'^2$$
$$= -\left[\frac{E - (v/c)F}{\sqrt{1 - v^2/c^2}}\right]^2 + \left[\frac{F - (v/c)E}{\sqrt{1 - v^2/c^2}}\right]^2 + \left[\frac{G + (v/c)Hh}{\sqrt{1 - v^2/c^2}}\right]^2 + \left[\frac{H - (v/c)Gh}{\sqrt{1 - v^2/c^2}}\right]^2$$
$$= -E^2 + F^2 + G^2 + H^2$$
$$= (Eh + Fj + Gj + Hk)(Eh - Fj - Gj - Hk)$$
$$= |P|^2.$$

Intermediate calculations are omitted.

The following three points are important regarding (17.1), (17.2), (17.3), and (17.4).

(1) The denominator is the constant $\sqrt{1 - v^2/c^2}$, which does not contain the variables t or x.
(2) The equation $|P| = |P'|$ is realized even if we multiply P and P' by the same constant.
(3) If the forms of (17.1), (17.2), (17.3), and (17.4) are used, any physical quantity becomes constant under coordinate transformation. Thus, a certain physical quantity must be written as (17.1), (17.2), (17.3), and (17.4) after proving by another method that it is constant by coordinate transformation. We must not claim that it is constant under coordinate transformation by using the previously written forms of (17.1), (17.2), (17.3), and (17.4).

In special relativity, velocity, acceleration, momentum, and force, which do not change their absolute values under coordinate transformation in four-dimensional space–time, are called four-velocity, four-acceleration, four-momentum, and four-force, respectively. After coordinate transformations, their denominators contain $\sqrt{1 - v^2/c^2}$ to make their absolute value constant, for reason (1). Moreover, for reason (2), their numerators are multiplied by the mass m_0, which is constant, and their denominators contain several factors of $\sqrt{1 - v^2/c^2}$ several times.

About (3), an explanation is required. For instance, it is assumed that the quantity of heat Q has the coordinate components $(Q_t h,\ Q_x i,\ Q_y j,\ Q_z k)$, and we write the formulae as (17.1), (17.2), (17.3), and (17.4), and using these formulae, we claim that the absolute value $|Q|$ is constant under coordinate transformation. The cause of the mistake in this theory is having written the formulae of (17.1), (17.2), (17.3), and (17.4) without proving the invariance of the absolute value $|Q|$ of the quantity of heat by another method. In this example, since Q is a scalar and it is clear that it does not have coordinate components, we recognize a mistake immediately. However, in vector quantities such as velocity or acceleration, it is easy to commit the above-mentioned mistake. A note is required.

17.2 Mass and coordinate transformations of velocity and momentum of the x-axial motion

In this section, using the new octonion, we examine whether velocity is constant under coordinate transformation. Suppose that observer B moves along the x-direction of stationary observer A with uniform velocity v. Moreover, the coordinates of the point mass D observed by A are $(cth,\ xi,\ yj,\ zk)$ and those by B are $(ct'h,\ x'i,\ y'j,\ z'k)$. At this time, the four-dimensional space–time diagram becomes that in Figure 17.1. Since the z-axis is not visible, it is drawn with a dashed line. Moreover, the velocity of D as observed by A is $V(V_t h,\ V_x i,\ V_y j,\ V_z k)$ and that by B is $V'(V_t' h,\ V_x' i,\ V_y' j,\ V_z' k)$. However, V and V' are new octonions and not vectors.

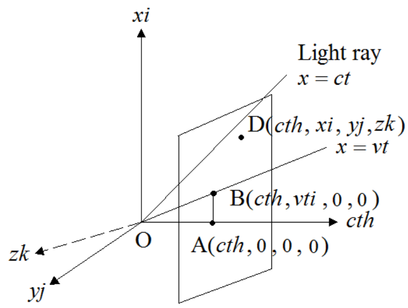


Figure 17.1

As explained in Section 10.5, the conversion equations of the velocities by the new Lorentz transformation are

$$V_x' = \frac{V_x - v}{1-(v/c^2)V_x}, \tag{10.25}$$

$$V_y' = \frac{V_y + (v/c)V_z h}{1-(v/c^2)V_x}, \tag{10.26}$$

$$V_z' = \frac{V_z - (v/c)V_y h}{1-(v/c^2)V_x}, \tag{10.27}$$

and the temporal component of the velocity V_t' is

$$V_t' = \frac{cdt'}{dt'} = c. \tag{17.5}$$

The square of the absolute value of V', including V_t', is

$$\begin{aligned} |V'|^2 &= (V_t' h + V_x' i + V_y' j + V_z' k)(V_t' h - V_x' i - V_y' j - V_z' k) \\ &= -V_t'^2 + V_x'^2 + V_y'^2 + V_z'^2. \end{aligned}$$

As explained in Section 17.1, since the denominators of (10.25), (10.26), (10.27), and (17.5) do not contain $\sqrt{1-v^2/c^2}$, it is obvious that $|V'|^2$ is not constant under coordinate transformation, i.e., $|V'|^2 \neq |V|^2$. Citing only the calculation result, the proof is as follows:

$$\begin{aligned} |V'|^2 &= -V_t'^2 + V_x'^2 + V_y'^2 + V_z'^2 \\ &= \frac{1-v^2/c^2}{(1-V_x v/c^2)^2}\left(-V_t^2 + V_x^2 + V_y^2 + V_z^2\right). \end{aligned}$$

To solve this issue, we assume that observer B approaches the point mass D, infinitesimally and V'_x along the x-direction of D with respect to B is 0. At that time, in the system of observer A, V_x along the x-direction of D and the velocity v of B are the same. That is,

$$V_x = v. \tag{17.6}$$

However, the velocities of B and D along the y- and z-axes need not be the same. Moreover, (17.6) can also be obtained by substituting $V'_x = 0$ into (10.25).

Substituting (17.6) into (10.26) and (10.27) yields

$$V'_y = \frac{V_y + (v/c)V_z h}{1 - v^2/c^2}, \tag{17.7}$$

$$V'_z = \frac{V_z - (v/c)V_y h}{1 - v^2/c^2}. \tag{17.8}$$

From (17.5), (17.6), (17.7), (17.8), $V'_x = 0$, and $V_t = cdt/dt = c$, we obtain

$$\begin{aligned}
|V'|^2 &= (V'_t h + V'_x i + V'_y j + V'_z k)(V'_t h - V'_x i - V'_y j - V'_z k) \\
&= -V'^2_t + V'^2_x + V'^2_y + V'^2_z \\
&= -c^2 + 0^2 + \left[\frac{V_y + (v/c)V_z h}{1 - v^2/c^2}\right]^2 + \left[\frac{V_z - (v/c)V_y h}{1 - v^2/c^2}\right]^2 \\
&= -c^2 + \frac{1}{(1 - v^2/c^2)^2} \\
&\quad \times \left[V_y^2 + 2(v/c)V_y V_z h + (v/c)^2 V_z^2 h^2 + V_z^2 - 2(v/c)V_y V_z h + (v/c)^2 V_y^2 h^2\right] \\
&= -c^2 + \frac{1}{(1 - v^2/c^2)^2}\left[V_y^2 - (v/c)^2 V_z^2 + V_z^2 - (v/c)^2 V_y^2\right] \\
&= -c^2 + \frac{1}{(1 - v^2/c^2)^2}(1 - v^2/c^2)(V_y^2 + V_z^2) \\
&= -c^2 + \frac{V_y^2 + V_z^2}{1 - v^2/c^2} \\
&= \frac{1}{1 - v^2/c^2}(-c^2 + v^2 + V_y^2 + V_z^2) \\
&= \frac{1}{1 - v^2/c^2}(-V_t^2 + V_x^2 + V_y^2 + V_z^2) \\
&= \frac{1}{1 - v^2/c^2}(V_t h + V_x i + V_y j + V_z k)(V_t h - V_x i - V_y j - V_z k) \\
&= \frac{1}{1 - v^2/c^2}|V|^2.
\end{aligned}$$

The square root is

$$|V'| = \frac{1}{\sqrt{1 - v^2/c^2}}\,|V|\,. \tag{17.9}$$

From (17.9), we can see that, in four-dimensional space–time, the absolute value of velocity is not constant under a coordinate transformation. Moreover, even if the time components $V_t = V_t' = c$ of velocity are not included in a calculation of the absolute value of velocity, $|V|$ does not become constant under a coordinate transformation. This can be easily proven by setting V_t and V_t' to 0 in the above calculations.

What can we understand from (17.9)? If both sides of (17.9) are multiplied by the rest mass m_0 of the point mass D, the equation becomes

$$m_0 \left| V' \right| = \frac{m_0}{\sqrt{1 - v^2/c^2}} \left| V \right| . \tag{17.10}$$

The rest mass m_0 is not the mass with respect to stationary observer A, but the mass with respect to the point mass D itself, or the moving observer B whose velocity along the x-direction is the same as that of D. That is, it is the mass in the coordinate system in which D is observed at rest. On the other hand, the mass with respect to observer A, who is moving with respcet to the point mass D, is the kinetic mass m. Substituting

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \tag{17.11}$$

into (17.10), we have

$$m_0 \left| V' \right| = m \left| V \right| . \tag{17.12}$$

The left-hand-side of (17.12) is the product of mass and velocity with respect to the observer B and the point mass D, who assume that the point mass D is at rest along the x-direction. The right-hand-side is the product of mass and velocity with respect to the observer A who is assumed to be moving with respect to the point mass D. Thus, (17.12) shows that the momentum of D is constant under a coordinate transformation. Moreover, from (17.11), we see that if the velocity v increases, the kinetic mass m increases, and if v approaches the velocity of light c, the mass becomes infinite.

The following two conclusions are obtained from the above results:

(1) In three-dimensional space or four-dimensional space–time, the absolute value of velocity is not constant under coordinate transformation.

(2) When the relation between the rest mass m_0 and the kinetic mass m is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

the momentum $m \left| V \right|$, including the temporal component, is constant under coordinate transformation.

Conclusion (2) can also be rewritten as follows:

(3) If the momentum $m\,|V|$, including the temporal component, is constant under coordinate transformation, the relation between the rest mass m_0 and the kinetic mass m is

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}}.$$

This result indicates that if we do not have mass variance through movement or invariance of momentum in axiom, we cannot obtain either law. Axiom is a law that is considered correct but cannot be proven. It is unknown which is an axiom. Moreover, there is a possibility that the two above laws are obtained from another axiom. At that time, they become theorems. We will discuss this possibility in Section 18.1.

Next, we examine whether

$$m_0\,|V'| = m\,|V| \tag{17.12}$$

is correct. If we rewrite the right-hand-side of (17.12) with the new octonion, from (17.6) and (17.11), we have

$$\frac{m_0}{\sqrt{1-v^2/c^2}}\left(ch + vi + V_y j + V_z k\right). \tag{17.13}$$

From (17.5), (17.7), (17.8), and $V'_x = 0$, the left-hand-side of (17.12) can de rewritten with the new octonion as

$$m_0\left\{ch + 0i + \left[\frac{V_y + (v/c)V_z h}{1-v^2/c^2}\right] j + \left[\frac{V_z - (v/c)V_y h}{1-v^2/c^2}\right] k\right\}. \tag{17.14}$$

If (17.13) becomes (17.14) under coordinate transformation, it is a proof of the correctness of (17.12). As explained in Section 17.1, we can perform the coordinate transformation by multiplying (17.13) by $\overline{B}/\,|B|$. The equations become

$$\begin{aligned}
&\frac{m_0}{\sqrt{1-v^2/c^2}}\left(ch + vi + V_y j + V_z k\right)\frac{(cth - vti)}{\sqrt{(cth+vti)(cth-vti)}}\\
&= \frac{m_0}{\sqrt{1-v^2/c^2}}\left(ch + vi + V_y j + V_z k\right)\frac{(cth - vti)}{cth\sqrt{1-(v/c)^2}}\\
&= \frac{m_0}{1-v^2/c^2}\left(ch + vi + V_y j + V_z k\right)(1 - vi/ch)\\
&= \frac{m_0}{1-v^2/c^2}\left(ch + vi + V_y j + V_z k\right)(1 + vhi/c)\\
&= \frac{m_0}{1-v^2/c^2}\left(ch + vi + V_y j + V_z k + vh^2 i + v^2 h i^2/c + V_y vhji/c + V_z vhki/c\right)\\
&= \frac{m_0}{1-v^2/c^2}\left(ch + vi + V_y j + V_z k - vi - v^2 h/c - V_y vhk/c + V_z vhj/c\right)
\end{aligned}$$

$$= \frac{m_0}{1 - v^2/c^2}\left(ch - v^2h/c + V_yj + V_zvhj/c + V_zk - V_yvhk/c\right)$$
$$= \frac{m_0}{1 - v^2/c^2}\left\{c(1 - v^2/c^2)h + [V_y + (v/c)V_zh]\, j + [V_z - (v/c)V_yh]\, k\right\}$$
$$= m_0\left\{ch + 0i + \left[\frac{V_y + (v/c)V_zh}{1 - v^2/c^2}\right] j + \left[\frac{V_z - (v/c)V_yh}{1 - v^2/c^2}\right] k\right\}.$$

Since this result is identical to (17.14), we have confirmed that the momentum

$$m_0\,|V'| = m\,|V| \qquad \left(\because\ m = \frac{m_0}{\sqrt{1 - v^2/c^2}}\right) \tag{17.15}$$

is constant under coordinate transformation using the new octonion. In (17.15), velocity v of observer B is the same as x-axial component V_x of the velocity of the point mass D. However, the y-axial component V_y and the z-axial component V_z have no limit. Thus, when the velocity v of B is in the x-direction of A, the change of the kinetic mass m of D is independent of V_y and V_z.

17.3 Mass and coordinate transformations of velocity and momentum of arbitrary axial motion

In this section, we investigate conservation of momentum and change of mass in the case where the velocity v of observer B is along an arbitrary direction in three-dimensional space. As explained in Section 10.8, when the velocity of observer B, who is moving along a straight line with uniform velocity as seen from static observer A, is $v(v_x,\ v_y,\ v_z)$, the general formulae of the new Lorentz transformation are

$$t' = \frac{1}{\sqrt{1 - v^2/c^2}}(t - v_xx/c^2 - v_yy/c^2 - v_zz/c^2), \tag{10.38}$$
$$x' = \frac{1}{\sqrt{1 - v^2/c^2}}(x - v_xt + v_zyh/c - v_yzh/c), \tag{10.39}$$
$$y' = \frac{1}{\sqrt{1 - v^2/c^2}}(y - v_yt + v_xzh/c - v_zxh/c), \tag{10.40}$$
$$z' = \frac{1}{\sqrt{1 - v^2/c^2}}(z - v_zt + v_yxh/c - v_xyh/c). \tag{10.41}$$

Infinitesimal components are

$$dt' = \frac{1}{\sqrt{1 - v^2/c^2}}(dt - v_xdx/c^2 - v_ydy/c^2 - v_zdz/c^2), \tag{17.16}$$
$$dx' = \frac{1}{\sqrt{1 - v^2/c^2}}(dx - v_xdt + v_zdyh/c - v_ydzh/c), \tag{17.17}$$

$$dy' = \frac{1}{\sqrt{1 - v^2/c^2}}(dy - v_y dt + v_x dzh/c - v_z dxh/c), \tag{17.18}$$

$$dz' = \frac{1}{\sqrt{1 - v^2/c^2}}(dz - v_z dt + v_y dxh/c - v_x dyh/c). \tag{17.19}$$

By dividing (17.17), (17.18), and (17.19) by (17.16), we find that

$$\begin{aligned}
V'_x &= \frac{dx'}{dt'} \\
&= \frac{dx - v_x dt + v_z dyh/c - v_y dzh/c}{dt - v_x dx/c^2 - v_y dy/c^2 - v_z dz/c^2} \\
&= \frac{dx/dt - v_x + (v_z h/c)dy/dt - (v_y h/c)dz/dt}{1 - (v_x/c^2)dx/dt - (v_y/c^2)dy/dt - (v_z/c^2)dz/dt} \\
&= \frac{V_x - v_x + (v_z h/c)V_y - (v_y h/c)V_z}{1 - (v_x/c^2)V_x - (v_y/c^2)V_y - (v_z/c^2)V_z}, \qquad (17.20) \\
V'_y &= \frac{dy'}{dt'} \\
&= \frac{V_y - v_y + (v_x h/c)V_z - (v_z h/c)V_x}{1 - (v_x/c^2)V_x - (v_y/c^2)V_y - (v_z/c^2)V_z}, \qquad (17.21) \\
V'_z &= \frac{dz'}{dt'} \\
&= \frac{V_z - v_z + (v_y h/c)V_x - (v_x h/c)V_y}{1 - (v_x/c^2)V_x - (v_y/c^2)V_y - (v_z/c^2)V_z}. \qquad (17.22)
\end{aligned}$$

Intermediate calculations were omitted. In addition, the temporal component of the velocity of D as seen from B is

$$V'_t = \frac{cdt'}{dt'} = c. \tag{17.23}$$

The absolute square of V', including V'_t, is

$$\begin{aligned}
|V'|^2 &= (V'_t h + V'_x i + V'_y j + V'_z k)(V'_t h - V'_x i - V'_y j - V'_z k) \\
&= -V'^2_t + V'^2_x + V'^2_y + V'^2_z.
\end{aligned}$$

As explained in Section 17.1, since the denominators of (17.20), (17.21), (17.22), and (17.23) do not contain $\sqrt{1 - v^2/c^2}$, it is obvious that $|V'|^2$ is not constant under coordinate transformation, i.e., $|V'|^2 \neq |V|^2$.

We now consider the case where B coincides with the point mass D. It is the same as that of the case where the velocity v of observer B is along the x-direction of the static observer A. Since the velocity $V(V_x,\ V_y,\ V_z)$ of D, as seen from A becomes the same as the velocity $v(v_x,\ v_y,\ v_z)$ of B, we can write

$$V_x = v_x,\ V_y = v_y,\ V_z = v_z. \tag{17.24}$$

Since the velocity $V'(V'_x,\ V'_y,\ V'_z)$ of D as seen from B is 0, we have

$$V'_x = V'_y = V'_z = 0. \tag{17.25}$$

From (17.23) and (17.25), the equation becomes

$$\begin{aligned} |V'|^2 &= -V_t'^2 + V_x'^2 + V_y'^2 + V_z'^2 \\ &= -c^2. \end{aligned} \tag{17.26}$$

From $V_t = cdt/dt = c,\ v^2 = v_x^2 + v_y^2 + v_z^2$, and (17.24), we find that

$$\begin{aligned} |V|^2 &= -V_t^2 + V_x^2 + V_y^2 + V_z^2 \\ &= -c^2 + v_x^2 + v_y^2 + v_z^2 \\ &= -c^2 + v^2 \\ &= -c^2(1 - v^2/c^2). \end{aligned} \tag{17.27}$$

From (17.26) and (17.27), we have

$$\begin{aligned} |V|^2 &= |V'|^2(1 - v^2/c^2), \\ |V'|^2 &= \frac{1}{1 - v^2/c^2}|V|^2. \end{aligned}$$

Taking the square root, the equation becomes

$$|V'| = \frac{1}{\sqrt{1 - v^2/c^2}}\,|V|\,.$$

Thus, even in four-dimensional space–time, the velocity is not constant under coordinate transformation. Thus, if both sides of this formula are multiplied by the rest mass m_0, we find that

$$m_0\,|V'| = \frac{m_0}{\sqrt{1 - v^2/c^2}}\,|V|\,.$$

If we assume the kinetic mass m is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

we have

$$m_0\,|V'| = m\,|V|\,. \tag{17.28}$$

The point mass D moves with velocity v along an arbitrary direction as seen from resting observer A. From (17.28), if the rest mass of D is assumed to be m_0 and the kinetic mass as m as seen from observer A, the kinetic mass is dependent on the uniform velocity v in an arbitrary direction, and the momentum is constant under coordinate transformation using the new octonion. The equation

$$m_0 \, |V'| = m \, |V| \qquad \left(: \; m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \right) \tag{17.15}$$

explained in Section 17.2, and (17.28) are the same. However, the velocity v of (17.15) is the velocity in the x-direction and the velocity v of (17.28) is the velocity that is composed of the x-, y-, and z-axial components.

17.4 Four-velocity and four-momentum

In the last section, it was proven that velocity is not constant under coordinate transformation in four-dimensional space–time. In special relativity, the velocity constant under coordinate transformation in four-dimensional space–time is called four-velocity. And when explaining four-velocity, it is written as follows. If the velocity is obtained using the proper time $\tau = t\sqrt{1 - v^2/c^2}$ instead of the time t of each coordinate system, the absolute value of velocity becomes constant under coordinate transformation. Thus, a static system and a kinetic system use the common time τ.

In this section, we prove that we can obtain the four-velocity of special relativity using the new octonion. As explained in Section 17.2, the velocity of point mass D, as seen from stationary observer A, is $V(V_t h, \; V_x i, \; V_y j, \; V_z k)$ and the velocity of D, as seen from observer B, who moves along a straight line in the x-direction of A with uniform velocity v, is $V'(V'_t h, \; V'_x i, \; V'_y j, \; V'_z k)$. When the velocity v of B and the velocity in the x-direction of D, i.e., V_x, are the same as seen from A, we have

$$|V'| = \frac{1}{\sqrt{1 - v^2/c^2}} \, |V| \, . \tag{17.9}$$

Thus, we cannot obtain a four-velocity that is constant under coordinate transformation. To solve this problem, although there is no rationale, V'_t, V'_x, V'_y, and V'_z are made using the proper time τ. At this time, to distinguish from V'_t, V'_x, V'_y, and V'_z calculated using dt', four-velocities obtained using $d\tau$ are written as U'_t, U'_x, U'_y, and U'_z. From the new Lorentz transformations

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{10.3}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \tag{10.4}$$

$$y' = \frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}}, \tag{10.5}$$

$$z' = \frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}}, \tag{10.6}$$

and $V_t = cdt/dt$, we have

$$\begin{aligned} U'_t &= \frac{cdt'}{d\tau} \\ &= \frac{cdt - c(v/c^2)dx}{\sqrt{1 - v^2/c^2}dt\sqrt{1 - v^2/c^2}} \\ &= \frac{cdt/dt - (v/c)dx/dt}{1 - v^2/c^2} \\ &= \frac{V_t - (v/c)V_x}{1 - v^2/c^2}, \end{aligned} \tag{17.29}$$

$$\begin{aligned} U'_x &= \frac{dx'}{d\tau} \\ &= \frac{dx - vdt}{\sqrt{1 - v^2/c^2}dt\sqrt{1 - v^2/c^2}} \\ &= \frac{dx/dt - (v/c)cdt/dt}{1 - v^2/c^2} \\ &= \frac{V_x - (v/c)V_t}{1 - v^2/c^2}, \end{aligned} \tag{17.30}$$

$$\begin{aligned} U'_y &= \frac{dy'}{d\tau} \\ &= \frac{dy + (v/c)dzh}{\sqrt{1 - v^2/c^2}dt\sqrt{1 - v^2/c^2}} \\ &= \frac{dy/dt + (v/c)(dz/dt)h}{1 - v^2/c^2} \\ &= \frac{V_y + (v/c)V_z h}{1 - v^2/c^2}, \end{aligned} \tag{17.31}$$

$$\begin{aligned} U'_z &= \frac{dz'}{d\tau} \\ &= \frac{dz - (v/c)dyh}{\sqrt{1 - v^2/c^2}dt\sqrt{1 - v^2/c^2}} \\ &= \frac{dz/dt - (v/c)(dy/dt)h}{1 - v^2/c^2} \\ &= \frac{V_z - (v/c)V_y h}{1 - v^2/c^2}. \end{aligned} \tag{17.32}$$

If the absolute value $|U'|$ of velocity U' of the point mass D is calculated using

(17.29), (17.30), (17.31), and (17.32), we find that

$$
\begin{aligned}
|U'|^2 &= (U'_t h + U'_x i + U'_y j + U'_z k)(U'_t h - U'_x i - U'_y j - U'_z k) \\
&= -U'^2_t + U'^2_x + U'^2_y + U'^2_z \\
&= -\left(\frac{V_t - (v/c)V_x}{1 - v^2/c^2}\right)^2 + \left(\frac{V_x - (v/c)V_t}{1 - v^2/c^2}\right)^2 \\
&\quad + \left(\frac{V_y + (v/c)V_z h}{1 - (v/c)^2}\right)^2 + \left(\frac{V_z - (v/c)V_y h}{1 - v^2/c^2}\right)^2 \\
&= \frac{1}{(1 - v^2/c^2)^2} \\
&\quad \times \left[-V_t^2 + 2(v/c)V_t V_x - (v/c)^2 V_x^2 + V_x^2 - 2(v/c)V_t V_x + (v/c)^2 V_t^2 \right. \\
&\qquad \left. + V_y^2 + 2(v/c)V_y V_z h - (v/c)^2 V_z^2 + V_z^2 - 2(v/c)V_z V_y h - (v/c)^2 V_y^2\right] \\
&= \frac{1}{(1 - v^2/c^2)^2} \\
&\quad \times \left[-\left(1 - v^2/c^2\right) V_t^2 + \left(1 - v^2/c^2\right) V_x^2 + \left(1 - v^2/c^2\right) V_y^2 + \left(1 - v^2/c^2\right) V_z^2\right] \\
&= \frac{1}{1 - v^2/c^2}\left(-V_t^2 + V_x^2 + V_y^2 + V_z^2\right) \\
&= \frac{1}{1 - v^2/c^2}|V|^2.
\end{aligned}
$$

Thus, if $c > v$, we can write

$$|U'| = \frac{1}{\sqrt{1 - v^2/c^2}}\,|V|\,. \tag{17.33}$$

However, U' on the left side of (17.33) is the velocity divided by the infinitesimal proper time $d\tau$ of the point mass D. V on the right-hand-side is the velocity divided by the infinitesimal time dt of static observer A.

As explained previously, in special relativity, the proper time τ of the point mass D is used instead of each observer's time t. All velocities are calculated using the proper time τ and are called four-velocities. Thus, if the velocity V of D, as seen from static observer A, is recalculated using the infinitesimal proper time $d\tau$ and the obtained four-velocity is written as U, the relation between U_x and V_x is

$$
\begin{aligned}
U_x &= \frac{dx}{d\tau} \\
&= \frac{dx}{dt\sqrt{1 - v^2/c^2}} \\
&= \frac{1}{\sqrt{1 - v^2/c^2}} V_x.
\end{aligned}
\tag{17.34}
$$

From the same method, $U_t,\ U_y$, and U_z become

$$U_t = \frac{1}{\sqrt{1-v^2/c^2}}V_t,\ U_y = \frac{1}{\sqrt{1-v^2/c^2}}V_y,\ U_z = \frac{1}{\sqrt{1-v^2/c^2}}V_z. \tag{17.35}$$

Thus, from (17.34) and (17.35), Equation (17.33) becomes

$$|U'| = |U|. \tag{17.36}$$

This indicates that four-velocity is constant under coordinate transformation. Moreover, if both sides of (17.36) are multiplied by the rest mass m_0 of D, we have

$$m_0\,|U'| = m_0\,|U|. \tag{17.37}$$

Here, if $m_0\,|U|$ is defined as four-momentum, it is constant under coordinate transformation. In addition, if (17.34) and (17.35) are applied to

$$U'_t = \frac{V_t-(v/c)V_x}{1-v^2/c^2}, \tag{17.29}$$

$$U'_x = \frac{V_x-(v/c)V_t}{1-v^2/c^2}, \tag{17.30}$$

$$U'_y = \frac{V_y+(v/c)V_z h}{1-v^2/c^2}, \tag{17.31}$$

$$U'_z = \frac{V_z-(v/c)V_y h}{1-v^2/c^2}, \tag{17.32}$$

we can have the formulae of the coordinate transformation of four-velocity, i.e.,

$$U'_t = \frac{U_t-(v/c)U_x}{\sqrt{1-v^2/c^2}},$$

$$U'_x = \frac{U_x-(v/c)U_t}{\sqrt{1-v^2/c^2}},$$

$$U'_y = \frac{U_y+(v/c)U_z h}{\sqrt{1-v^2/c^2}},$$

$$U'_z = \frac{U_z-(v/c)U_y h}{\sqrt{1-v^2/c^2}}.$$

The above verification in special relativity looks logical. However, it is not mathematical. The reason is that since the result is not contradictory, the proper time τ is used. Using the theory that is assumed correct since the desired result is obtained, although the reason is unknown, we understand only a part of the reality.

As explained in Section 17.1, if the formulae

$$E' = \frac{E-(v/c)F}{\sqrt{1-v^2/c^2}}, \tag{17.1}$$

$$F' = \frac{F - (v/c)E}{\sqrt{1 - v^2/c^2}}, \tag{17.2}$$

$$G' = \frac{G + (v/c)Hh}{\sqrt{1 - v^2/c^2}}, \tag{17.3}$$

$$H' = \frac{H - (v/c)Gh}{\sqrt{1 - v^2/c^2}} \tag{17.4}$$

are used, any physical quantity becomes constant under coordinate transformation. Thus, after proving that a certain physical quantity is constant under coordinate transformation using another method, we must put it into the form of (17.1), (17.2), (17.3), and (17.4). It is not logical that after making the transformation formulae of four-velocity, i.e., (17.1), (17.2), (17.3), and (17.4), using the proper time τ, the invariance under coordinate transformation is asserted based on these formulae. Thus, in the next section, we explain the defects of four-velocity.

17.5 Defects of four-velocity

In Section 17.2, we proved that in four-dimensional space–time, velocity is not constant under coordinate transformation. However, in the last section, we proved that four-velocity is constant under coordinate transformation. Comparing the two conclusions, we find that the theory of four-velocity and four-momentum has the following four defects.

(1) In the theory of four-momentum, mass does not change by movement.

(2) In order to make four-velocity constant under coordinate transformation, velocity is calculated using the proper time τ. Thus, the coordinate transformation invariance of four-velocity is not proven.

(3) The direction of the velocity v of observer B is in the x-direction of observer A, and the direction of the velocity of the point mass D is in an arbitrary direction in four-dimensional space–time. However, the case where B and D are in agreement is considered.

(4) For observer B moving together with the point mass D, the velocity of D must be 0. However, velocity $U(U'_t,\ U'_x,\ U'_y,\ U'_z)$ of D observed by B is considered.

Next, we examine the above-mentioned defects of four-velocity and four-momentum.

Defect (1)

From

$$m_0 \left|U'\right| = m_0 \left|U\right|, \tag{17.37}$$

mass is always constant. It is proven in special relativity that mass changes according to velocity. Thus, it is considered that four-velocity is an expedient concept that does not express the property of space–time.

Defect (2)

As explained in Section 17.1, the coordinate transformation invariance of four-velocity must be proved by another method besides by the formulae of coordinate transformation. In order to prove that there is no mathematical basis for using the proper time τ of the point mass D in calculations of all velocities, we prove that by using time t of stationary observer A instead of the proper time τ, four-velocity can be made constant under coordinate transformation. As explained in Section 10.5, the infinitesimal quantities of the new Lorentz transformations are

$$dt' = \frac{dt - (v/c^2)dx}{\sqrt{1 - v^2/c^2}}, \tag{10.21}$$

$$dx' = \frac{dx - vdt}{\sqrt{1 - v^2/c^2}}, \tag{10.22}$$

$$dy' = \frac{dy + (v/c)dzh}{\sqrt{1 - v^2/c^2}}, \tag{10.23}$$

$$dz' = \frac{dz - (v/c)dyh}{\sqrt{1 - v^2/c^2}}. \tag{10.24}$$

If (10.21), (10.22), (10.23), and (10.24) are divided by the infinitesimal time dt of observer A, we have

$$\begin{aligned} V_t' &= \frac{cdt'}{dt} \\ &= \frac{cdt - (v/c)dx}{dt\sqrt{1 - v^2/c^2}} \\ &= \frac{c - (v/c)dx/dt}{\sqrt{1 - v^2/c^2}} \\ &= \frac{c - (v/c)V_x}{\sqrt{1 - v^2/c^2}}, \end{aligned} \tag{17.38}$$

$$\begin{aligned} V_x' &= \frac{dx'}{dt} \\ &= \frac{dx - vdt}{dt\sqrt{1 - v^2/c^2}} \\ &= \frac{dx/dt - v}{\sqrt{1 - v^2/c^2}} \\ &= \frac{V_x - v}{\sqrt{1 - v^2/c^2}}, \end{aligned} \tag{17.39}$$

$$
\begin{aligned}
V'_y &= \frac{dy'}{dt} \\
&= \frac{dy + (v/c)dzh}{dt\sqrt{1 - v^2/c^2}} \\
&= \frac{dy/dt + (v/c)\ (dz/dt)h}{\sqrt{1 - v^2/c^2}} \\
&= \frac{V_y + (v/c)V_z h}{\sqrt{1 - v^2/c^2}}, \qquad (17.40) \\
V'_z &= \frac{dz'}{dt} \\
&= \frac{dz - (v/c)dyh}{dt\sqrt{1 - v^2/c^2}} \\
&= \frac{dz/dt - (v/c)(dy/dt)h}{\sqrt{1 - v^2/c^2}t} \\
&= \frac{V_z - (v/c)V_y h}{\sqrt{1 - v^2/c^2}}. \qquad (17.41)
\end{aligned}
$$

Next, if we calculate $|V'|^2$ using (17.38), (17.39), (17.40), and (17.41), we find that

$$
\begin{aligned}
|V'|^2 &= (V'_t h + V'_x i + V'_y j + V'_z k)(V'_t h - V'_x i - V'_y j - V'_z k) \\
&= -V'^2_t + V'^2_x + V'^2_y + V'^2_z \\
&= -\left[\frac{c - (v/c)V_x}{\sqrt{1 - v^2/c^2}}\right]^2 + \left[\frac{V_x - v}{\sqrt{1 - v^2/c^2}}\right]^2 \\
&\quad + \left[\frac{V_y + (v/c)V_z h}{\sqrt{1 - v^2/c^2}}\right]^2 + \left[\frac{V_z - (v/c)V_y h}{\sqrt{1 - v^2/c^2}}\right]^2 \\
&= \frac{1}{1 - v^2/c^2} \\
&\quad \times \left[-c^2 + 2c(v/c)V_x - (v/c)^2 V_x^2 + V_x^2 - 2vV_x + v^2\right. \\
&\qquad \left. + V_y^2 + 2(v/c)V_y V_z h - (v/c)^2 V_z^2 + V_z^2 - 2(v/c)V_z V_y h - (v/c)^2 V_y^2\ \right] \\
&= \frac{1}{1 - v^2/c^2}\left[-c^2 - (v/c)^2 V_x^2 + V_x^2 + v^2 + V_y^2 - (v/c)^2 V_z + V_z^2 - (v/c)^2 V_y^2\right] \\
&= \frac{1}{1 - v^2/c^2}\left[-c^2(1 - v^2/c^2) + V_x^2(1 - v^2/c^2) + (V_y^2 + V_z^2)(1 - v^2/c^2)\right] \\
&= -c^2 + V_x^2 + V_y^2 + V_z^2.
\end{aligned}
$$

Since $V_t = cdt/dt = c$, the above equation becomes

$$
|V'|^2 = -V_t^2 + V_x^2 + V_y^2 + V_z^2
$$

$$= (V_t h + V_x i + V_y j + V_z k)(V_t h - V_x i - V_y j - V_z k)$$
$$= |V|^2.$$

Thus, we have

$$|V'| = |V|.$$

From this formula, we see that even if we calculate all velocities using the time t of stationary observer A instead of the proper time τ of the point mass D, velocities become constant under coordinate transformation. That is, there is no reason for having to use the proper time τ when we obtain the four-velocity.

As explained in Section 17.1, if the denominator of the formula of a certain physical quantity has a factor of $\sqrt{1 - v^2/c^2}$, the physical quantity becomes constant under coordinate transformation. Since the formula for the velocity obtained using the proper time τ of the point mass D, and the formula for the velocity obtained using the time t of stationary observer A, both have $\sqrt{1 - v^2/c^2}$ in the denominator, they become constant under coordinate transformation. Without proving the coordinate transformation invariance using another method, the invariance of the coordinate transformation of velocity and four-momentum are asserted by using the formula that has $\sqrt{1 - v^2/c^2}$ in the denominator. Thus, we have the contradictory result that when using four-velocity, mass does not change by movement.

Defect (3)

The Lorentz transformation of special relativity does not have general formulae for the case of the velocity of observer B being in an arbitrary direction. The direction of velocity of B is always in the x-direction of stationary observer A. The y- and z-axial components of the velocity of observer B do not exist. However, the velocity of the point mass D has y- and z-axial components. In the process in which the Lorentz transformation is applied to four-velocity, the direction of the velocity v changes to an arbitrary direction without verification. On the other hand, in Section 17.3, we proved correct the formulae of four-velocity, in the case of velocity v of observer B being in an arbitrary direction, using the new Lorentz transformation.

Defect (4)

In special relativity, it is assumed that although observer B is approaching the point mass D, infinitesimally, B is not in accordance. Since they are not in agreement, it is assumed that four coordinate components of the velocity of D can be observed by B. However, since B is in agreement with D, the proper time τ of D is used when finding the velocity. Thus, this theory is not mathematical. Moreover, velocity is obtained by assuming that according to observer B moving with the point mass D, D moves an infinitesimal distance $ds(cdt,\ dx,\ dy,\ dz)$ at a certain moment. This theory does not have a clear basis either.

As shown in Section 17.3, from a purely mathematical standpoint, velocity is not constant under coordinate transformation but momentum is. At that time, mass varies with velocity. The reason for the existence of coordinate transformation invariance of momentum, the change of mass according to velocity, and the meaning of the temporal component of velocity, will be discussed in Section 18.1. From now on, in this book, when velocity and acceleration are calculated, we use proper time τ for point masses and time t for the observer. Only in the case of four-velocity, four-acceleration, four-momentum, and four-force of special relativity, the proper time τ is used for calculations of all velocities.

17.6 Conservation of momentum

In Section 17.3, we explained the coordinate transformation invariance of the momentum. In this section, we explain a law of conservation of momentum. The coordinate transformation invariance of momentum means that the absolute value of momentum is constant in any coordinate system. On the other hand, the law of conservation of momentum means that the sum of each coordinate component of momentum is constant before and after movement. Although two terms are alike, semantics differ. The law of conservation of momentum is proven by special relativity. In this section, the law of conservation of momentum is proved using the new octonion. Moreover, although the case where a point mass and an observer move together in three-dimensional space is treated in special relativity, using the new octonion, we consider the case where the velocity in the x-direction of a point mass and an observer's velocity are in agreement. However, in order to simplify, the imaginary numbers $i,\ j$, and k are not used, but the imaginary number h is used when the component in the negative world is expressed.

As shown in Figure 17.2, it is assumed that a point mass A with rest mass M_0 is at rest at origin O, which is the origin of the x-, y-, and z-coordinates of resting observer O. The z-axis is omitted. Next suppose that A is divided into point masses B and D with rest masses m_0 by an internal explosion. It is assumed that the fission takes place in the x-y plane. If the velocity of B is $v(v_x,\ v_y,\ 0)$ at that time, the velocity of D is $-v(-v_x,\ -v_y,\ 0)$ by the symmetry property. In addition, suppose that observer O' who moves along the x-axis at the same velocity as v_x in

x-direction of B. The coordinates axes of O' are the x'-, y'-, z'-axes. However, the z'-axis is omitted in Figure 17.2.

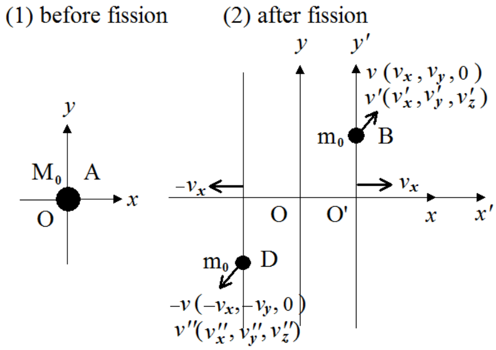


Figure 17.2

Since M_0 and m_0 are rest masses, we assume the kinetic masses to be M and m, respectively. As explaned in Section 17.2, the relation between kinetic mass and rest mass is

$$M = \frac{M_0}{\sqrt{1 - v^2/c^2}},$$
$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}. \tag{17.11}$$

However, v is a three-dimensional space velocity and is not four-velocity.

First, we prove the law of conservation of momentum as seen by stationary observer O. The velocity of A before the fission is $(0,\ 0,\ 0)$. The velocity of B after the fission is $v(v_x,\ v_y,\ 0)$ and the velocity of D is $-v(-v_x,\ -v_y,\ -0)$. Therefore, the sum of the momenta of the x-, y-, and z-axes before and after the fission are calculated by

$$M_0 \times 0 = \frac{m_0}{\sqrt{1 - (v_x^2 + v_y^2)/c^2}} v_x + \frac{m_0}{\sqrt{1 - (v_x^2 + v_y^2)/c^2}} (-v_x),$$
$$M_0 \times 0 = \frac{m_0}{\sqrt{1 - (v_x^2 + v_y^2)/c^2}} v_y + \frac{m_0}{\sqrt{1 - (v_x^2 + v_y^2)/c^2}} (-v_y),$$
$$M_0 \times 0 = \frac{m_0}{\sqrt{1 - (v_x^2 + v_y^2)/c^2}} \times 0 + \frac{m_0}{\sqrt{1 - (v_x^2 + v_y^2)/c^2}} \times 0.$$

Thus, the law of conservation of momentum is realized about the x-, y-, and z-components.

Next, we investigate whether or not the law of conservation of momentum is realized as seen by observer O' who moves in the x-direction with velocity v_x. As explained in Section 10.5, if the velocity of a certain point mass is observed as $V(V_x,\ V_y,\ V_z)$ by stationary observer O, and as $V'(V'_x,\ V'_y,\ V'_z)$ by observer O', who is moving in the x-direction with velocity v_x, the relation between V and V' is

$$V'_x = \frac{V_x - v_x}{1-(v_x/c^2)V_x}, \tag{10.25}$$

$$V'_y = \frac{V_y + (v_x/c)V_z h}{1-(v_x/c^2)V_x}, \tag{10.26}$$

$$V'_z = \frac{V_z - (v_x/c)V_y h}{1-(v_x/c^2)V_x}. \tag{10.27}$$

In Figure 17.2, the velocity of D as seen by O' is $v''(v''_x,\ v''_y,\ v''_z)$. Since the velocity of D as seen by O is $-v(-v_x,\ -v_y,\ 0)$, by replacing $V_x \to -v_x,\ V_y \to -v_y,\ V_z \to 0,\ V'_x \to v''_x,\ V'_y \to v''_y$, and $V'_z \to v''_z$ in (10.25), (10.26), and (10.27), we find that

$$\begin{aligned} v''_x &= \frac{-v_x - v_x}{1+(v_x/c^2)v_x} \\ &= \frac{-2v_x}{1+v_x^2/c^2}, \end{aligned} \tag{17.42}$$

$$v''_y = \frac{-v_y}{1+v_x^2/c^2}, \tag{17.43}$$

$$v''_z = \frac{(v_x v_y/c)h}{1+v_x^2/c^2}. \tag{17.44}$$

From (17.42), (17.43), and (17.44), we have

$$\begin{aligned} v''^2 &= v''^2_x + v''^2_y + v''^2_z \\ &= \frac{4v_x^2 + v_y^2 - v_x^2 v_y^2/c^2}{(1+v_x^2/c^2)^2}. \end{aligned} \tag{17.45}$$

Then, the velocity of B as seen by O' is $v'(v'_x,\ v'_y,\ v'_z)$. Since the velocity of B as seen by O is $v(v_x,\ v_y,\ 0)$, by replacing $V_x \to v_x,\ V_y \to v_y,\ V_z \to 0,\ V'_x \to v'_x,\ V'_y \to v'_y$, and $V'_z \to v'_z$ in (10.25), (10.26), and (17.27), we find that

$$\begin{aligned} v'_x &= \frac{v_x - v_x}{1-v_x^2/c^2} \\ &= 0, \end{aligned} \tag{17.46}$$

$$v'_y = \frac{v_y}{1-v_x^2/c^2}, \tag{17.47}$$

$$v'_z = \frac{-(v_x v_y/c)h}{1-v_x^2/c^2}. \tag{17.48}$$

From (17.46), (17.47), and (17.48), we have

$$
\begin{aligned}
v'^2 &= v_x'^2 + v_y'^2 + v_z'^2 \\
&= \frac{v_y^2 - v_x^2 v_y^2/c^2}{(1 - v_x^2/c^2)^2} \\
&= \frac{v_y^2(1 - v_x^2/c^2)}{(1 - v_x^2/c^2)^2} \\
&= \frac{v_y^2}{1 - v_x^2/c^2}.
\end{aligned} \tag{17.49}
$$

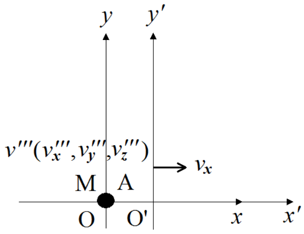


Figure 17.3

In addition, as shown in Figure 17.3, we assume that the velocity of A from O' before the fission is $v'''(v_x''',\ v_y''',\ v_z''')$. Since the velocity of A as seen by O is $v(0,\ 0,\ 0)$, by replacing $V_x \to 0$, $V_y \to 0$, $V_z \to 0$, $V_x' \to v_x'''$, $V_y' \to v_y'''$, and $V_z' \to v_z'''$ in (10.25), (10.26), and (10.27), we find that

$$v_x''' = -v_x, \tag{17.50}$$

$$v_y''' = 0, \tag{17.51}$$

$$v_z''' = 0. \tag{17.52}$$

From (17.50), (17.51), and (17.52), we have

$$
\begin{aligned}
v'''^2 &= v_x'''^2 + v_y'''^2 + v_z'''^2 \\
&= v_x^2.
\end{aligned} \tag{17.53}
$$

If (17.45), (17.49), and (17.53) are written collectively, we have

$$v''^2 = \frac{4v_x^2 + v_y^2 - v_x^2 v_y^2/c^2}{(1 + v_x^2/c^2)^2}, \tag{17.45}$$

$$v'^2 = \frac{v_y^2}{1 - v_x^2/c^2}, \tag{17.49}$$

$$v'''^2 = v_x^2. \tag{17.53}$$

Using these velocities, we can calculate the kinetic mass of the law of conservation of momentum below.

Before examining the momentum conservation law in the x-direction as seen by O', we examine the momentum conservation law in the y- and z-directions. The momentum in the y-direction of A before the fission as seen by O' is p_y''', and the momenta in the y-direction of B and D after the fission are p_y' and p_y'', respectively. If we replace v^2 in the general formula of momentum, i.e.,

$$p_y = \frac{m_0 v_y}{\sqrt{1 - v^2/c^2}},$$

by

$$v'^2 = \frac{v_y^2}{1 - v_x^2/c^2} \tag{17.49}$$

and replace v_y with the previously obtained expression

$$v_y' = \frac{v_y}{1 - v_x^2/c^2}, \tag{17.47}$$

we find that

$$\begin{aligned}
p_y' &= \frac{m_0}{\sqrt{1 - \left[v_y^2/(1 - v_x^2/c^2)\right]/c^2}} \times \frac{v_y}{1 - v_x^2/c^2} \\
&= \frac{m_0 v_y}{\sqrt{(1 - v_x^2/c^2)^2 - (1 - v_x^2/c^2)^2 \left[v_y^2/(1 - v_x^2/c^2)\right]/c^2}} \\
&= \frac{m_0 v_y}{\sqrt{(1 - v_x^2/c^2)^2 - (1 - v_x^2/c^2) v_y^2/c^2}} \\
&= \frac{m_0 v_y}{\sqrt{(1 - v_x^2/c^2)\left[(1 - v_x^2/c^2) - v_y^2/c^2\right]}} \\
&= \frac{m_0 v_y}{\sqrt{(1 - v_x^2/c^2)(1 - v_x^2/c^2 - v_y^2/c^2)}}.
\end{aligned} \tag{17.54}$$

Similarly, from the previously obtained expression

$$v_y'' = \frac{-v_y}{1 + v_x^2/c^2}, \tag{17.43}$$

$$v''^2 = \frac{4v_x^2 + v_y^2 - v_x^2 v_y^2/c^2}{(1 + v_x^2/c^2)^2}, \tag{17.45}$$

we find that

$$p_y'' = \frac{m_0}{\sqrt{1 - \left[(4v_x^2 + v_y^2 - v_x^2 v_y^2/c^2)/(1 + v_x^2/c^2)^2\right]/c^2}} \times \frac{-v_y}{1 + v_x^2/c^2}$$

$$= \frac{-m_0 v_y}{\sqrt{(1+v_x^2/c^2)^2 - (4v_x^2/c^2 + v_y^2/c^2 - v_x^2 v_y^2/c^4)}}$$

$$= \frac{-m_0 v_y}{\sqrt{1+2v_x^2/c^2 + v_x^4/c^4 - 4v_x^2/c^2 - v_y^2/c^2 + v_x^2 v_y^2/c^4}}$$

$$= \frac{-m_0 v_y}{\sqrt{(1-v_x^2/c^2)^2 - v_y^2/c^2(1-v_x^2/c^2)}}$$

$$= \frac{-m_0 v_y}{\sqrt{(1-v_x^2/c^2)(1-v_x^2/c^2-v_y^2/c^2)}}. \tag{17.55}$$

Similarly, from the previously obtained expression

$$v_y''' = 0, \tag{17.51}$$

$$v'''^2 = v_x^2, \tag{17.53}$$

we find that

$$p_y''' = \frac{M_0}{\sqrt{1-v_x^2/c^2}} \times 0$$

$$= 0. \tag{17.56}$$

From (17.54), (17.55), and (17.56), we have

$$p_y' + p_y'' = 0 = p_y'''.$$

Thus, the momentum conservation law in the y-direction is realized. Since the equation of the momentum in the z-direction becomes the formula obtained by replacing v_y of the numerator of (17.54), and (17.55) by $-v_x v_y h/c$, we can write

$$p_z' + p_z'' = 0 = p_z'''.$$

Thus, the momentum conservation law in the z-direction is also realized.

Next, we investigate the momentum conservation law in the x-direction. The momentum in the x-direction of A before the fission as seen by O' is p_x''', and the momenta in the x-direction of B and D after the fission are p_x' and p_x'', respectively. From the previously obtained expression

$$v_x' = 0, \tag{17.46}$$

$$v'^2 = \frac{v_y^2}{1-v_x^2/c^2}, \tag{17.49}$$

we find that

$$
\begin{aligned}
p'_x &= \frac{m_0}{\sqrt{1 - v'^2/c^2}} \times v'_x \\
&= \frac{m_0}{\sqrt{1 - \left[v_y^2/(1 - v_x^2/c^2)\right]/c^2}} \times 0 \\
&= 0.
\end{aligned} \tag{17.57}
$$

Similarly, from the previously obtained expression

$$
v''_x = \frac{-2v_x}{1 + v_x^2/c^2}, \tag{17.42}
$$

$$
v''^2 = \frac{4v_x^2 + v_y^2 - v_x^2 v_y^2/c^2}{(1 + v_x^2/c^2)^2}, \tag{17.45}
$$

we find that

$$
\begin{aligned}
p''_x &= \frac{m_0}{\sqrt{1 - \left[(4v_x^2 + v_y^2 - v_x^2 v_y^2/c^2)/(1 + v_x^2/c^2)^2\right]/c^2}} \times \frac{-2v_x}{1 + v_x^2/c^2} \\
&= \frac{-2m_0 v_x}{\sqrt{(1 + v_x^2/c^2)^2 - (4v_x^2/c^2 + v_y^2/c^2 - v_x^2 v_y^2/c^4)}} \\
&= \frac{-2m_0 v_x}{\sqrt{1 + 2v_x^2/c^2 + v_x^4/c^4 - 4v_x^2/c^2 - v_y^2/c^2 + v_x^2 v_y^2/c^4}} \\
&= \frac{-2m_0 v_x}{\sqrt{(1 - v_x^2/c^2)^2 - v_y^2/c^2(1 - v_x^2/c^2)}} \\
&= \frac{-2m_0 v_x}{\sqrt{(1 - v_x^2/c^2)(1 - v_x^2/c^2 - v_y^2/c^2)}}.
\end{aligned} \tag{17.58}
$$

Similarly, from the previously obtained expression

$$
v'''_x = -v_x, \tag{17.50}
$$

$$
v'''^2 = v_x^2, \tag{17.53}
$$

we find that

$$
\begin{aligned}
p'''_x &= \frac{M_0}{\sqrt{1 - v_x^2/c^2}} \times (-v_x) \\
&= \frac{-M_0 v_x}{\sqrt{1 - v_x^2/c^2}}.
\end{aligned} \tag{17.59}
$$

From (17.57), (17.58), and (17.59), we have

$$
p'_x + p''_x - p'''_x = \frac{-2m_0 v_x}{\sqrt{(1 - v_x^2/c^2)(1 - v_x^2/c^2 - v_y^2/c^2)}} - \frac{-M_0 v_x}{\sqrt{1 - v_x^2/c^2}}. \tag{17.60}
$$

We now consider what is necessary for (17.60) to become 0. If we assume

$$p'_x + p''_x - p'''_x = 0,$$

from (17.60), we have

$$\frac{2m_0 v_x}{\sqrt{(1 - v_x^2/c^2)(1 - v_x^2/c^2 - v_y^2/c^2)}} = \frac{M_0 v_x}{\sqrt{1 - v_x^2/c^2}},$$

$$\frac{2m_0}{\sqrt{1 - v_x^2/c^2 - v_y^2/c^2}} = M_0. \qquad (17.61)$$

Since the velocity v of point mass B is $v = v_x^2 + v_y^2 + 0$, (17.61) becomes

$$\frac{2m_0}{\sqrt{1 - v^2/c^2}} = M_0. \qquad (17.62)$$

When v is infinitesimal compared to c, from

$$\frac{1}{\sqrt{1 - v^2/c^2}} \doteqdot 1 - (v^2/c^2)(-1/2)$$
$$= 1 + (v^2/c^2)/2,$$

(17.62) becomes

$$2m_0 + 2m_0(v^2/c^2)/2 \doteqdot M_0.$$

By multiplying both side by c^2, the equation becomes

$$M_0 c^2 \doteqdot 2m_0 c^2 + 2m_0 v^2/2. \qquad (17.63)$$

As proven in Einstein's special relativity, if the rest mass m_0 is changed into energy E, we have

$$E = m_0 c^2.$$

The left-hand-side of (17.63) expresses the rest energy of A before fission. The first part of the right-hand-side expresses the sum of the rest energies of B and D. The second part expresses the sum of the kinetic energies of the Newtonian mechanics of B and D. That is, (17.63) indicates that the rest energy of the point mass A changes to sum of the rest energies and kinetic energies of the point masses B and D. The conclusion obtained from this verification is the same as the conclusion of special relativity. This shows that a contradiction does not occur in calculations using the new octonion.

In the above discussion, velocities are not the four-velocities using the proper time τ but the velocities using the time t of each observer. The reason is that if four-velocity is used, mass does not change according to the velocity. At that time, we cannot conclude that the rest mass M_0 becomes two times the rest mass m_0, and that the energy becomes two times the kinetic energy $m_0v^2/2$ after fission. Also from this, it is concluded that four-velocity does not represent a real physical phenomenon and is the quantity invented for the convenience of calculations. In addition, there may be some readers who assume that the energy added before fission converts into kinetic energy after fission. However, even if we use gunpowder to trigger fission, the mass of the gunpowder is contained in the rest mass M_0, and we can consider that mass changed to energy. The energy used for fission is not added from external sources.

17.7 Acceleration and four-acceleration

In this section, the acceleration as seen by the observer who is undergoing uniform motion is considered. Calculations are performed by the new octonion.

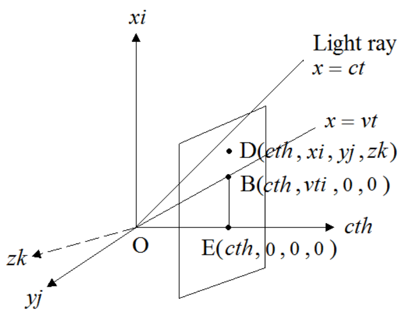


Figure 17.4

In Figure 17.4, observer B is moving in the x-direction of static observer E with a constant velocity v. The point mass D as seen by B is exerting the force that causes the acceleration movement. When B approaches and meets with D, B cannot measure the acceleration of D. This is because acceleration cannot be measured by oneself. The famous Gedanken experiment of general relativity has a freely falling elevator. The idea is that in the elevator, whose tie is broken and is undergoing free fall, a person inside cannot feel any gravitational acceleration because the position between the elevator and the person does not change. This idea is termed equivalence principle in general relativity. However, in special relativity, an observer measures its own acceleration, which is called proper acceleration. To solve this contradiction, in special relativity, the case where B infinitesimaly aproaches D is not considered, whereas the case in which B, moving together with D, stands still at a certain moment and measures the acceleration of D is considered. This is called an instantaneous static system. However, in this book, the case where B infinitesimaly aproaches D is considered. At that time, the time t' infinitesimaly approaches the proper time $\tau = t\sqrt{1 - v^2/c^2}$ of D. Thus, we use $d\tau = dt\sqrt{1 - v^2/c^2}$ instead of dt'.

Since the new Lorentz transformations are

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \tag{10.3}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \tag{10.4}$$

$$y' = \frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}}, \tag{10.5}$$

$$z' = \frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}}, \tag{10.6}$$

the infinitesimal quantities are

$$dt' = \frac{dt - (v/c^2)dx}{\sqrt{1 - v^2/c^2}}, \tag{10.21}$$

$$dx' = \frac{dx - vdt}{\sqrt{1 - v^2/c^2}}, \tag{10.22}$$

$$dy' = \frac{dy + (v/c)dzh}{\sqrt{1 - v^2/c^2}}, \tag{10.23}$$

$$dz' = \frac{dz - (v/c)dyh}{\sqrt{1 - v^2/c^2}}. \tag{10.24}$$

To obtain the proper velocities, we divide (10.21), (10.22), (10.23), and (10.24) by infinitesimal proper time $d\tau = dt\sqrt{1 - v^2/c^2}$, and find that

$$\begin{aligned} V'_t &= \frac{cdt'}{d\tau} \\ &= \frac{cdt - c(v/c^2)dx}{\sqrt{1 - v^2/c^2}dt\sqrt{1 - v^2/c^2}} \\ &= \frac{cdt/dt - (v/c)dx/dt}{1 - v^2/c^2} \\ &= \frac{c - (v/c)V_x}{1 - v^2/c^2}, \end{aligned} \tag{17.64}$$

$$\begin{aligned} V'_x &= \frac{dx'}{d\tau} \\ &= \frac{dx - vdt}{\sqrt{1 - v^2/c^2}dt\sqrt{1 - v^2/c^2}} \end{aligned}$$

$$= \frac{dx/dt - (v/c)cdt/dt}{1 - v^2/c^2}$$

$$= \frac{V_x - v}{1 - v^2/c^2}, \tag{17.65}$$

$$V'_y = \frac{dy'}{d\tau}$$

$$= \frac{dy + (v/c)dzh}{\sqrt{1 - v^2/c^2}dt\sqrt{1 - v^2/c^2}}$$

$$= \frac{dy/dt + (v/c)\ hdz/dt}{1 - v^2/c^2}$$

$$= \frac{V_y + (v/c)V_z h}{1 - v^2/c^2}, \tag{17.66}$$

$$V'_z = \frac{dz'}{d\tau}$$

$$= \frac{dz - (v/c)dyh}{\sqrt{1 - v^2/c^2}dt\sqrt{1 - v^2/c^2}}$$

$$= \frac{dz/dt - (v/c)\ hdy/dt}{1 - v^2/c^2}$$

$$= \frac{V_z - (v/c)V_y h}{1 - v^2/c^2}. \tag{17.67}$$

However, V_x, V_y, and V_z are the velocities found using the infinitesimal time dt of observer E.

In the case where observer B approaches point mass D, infinitesimally, the acceleration a' of D is also calculated by dividing V' by the proper time τ. However, the acceleration of D as seen by static observer E is obtained by dividing V by the time t, and is written as a. From (17.64), (17.65), (17.66), and (17.67), we find that

$$a'_t = \frac{dV'_t}{d\tau}$$

$$= \frac{0 - (v/c)dV_x}{(1 - v^2/c^2)dt\sqrt{1 - v^2/c^2}}$$

$$= \frac{-(v/c)dV_x/dt}{(1 - v^2/c^2)^{3/2}}$$

$$= \frac{-(v/c)a_x}{(1 - v^2/c^2)^{3/2}}, \tag{17.68}$$

$$a'_x = \frac{dV'_x}{d\tau}$$

$$= \frac{dV_x - 0}{(1 - v^2/c^2)dt\sqrt{1 - v^2/c^2}}$$

$$= \frac{dV_x/dt}{(1 - v^2/c^2)^{3/2}}$$

$$= \frac{a_x}{(1 - v^2/c^2)^{3/2}}, \tag{17.69}$$

$$a'_y = \frac{dV'_y}{d\tau}$$

$$= \frac{dV_y + (v/c)hdV_z}{(1 - v^2/c^2)dt\sqrt{1 - v^2/c^2}}$$

$$= \frac{dV_y/dt + (v/c)\ hdV_z/dt}{(1 - v^2/c^2)^{3/2}}$$

$$= \frac{a_y + (v/c)a_z h}{(1 - v^2/c^2)^{3/2}}, \tag{17.70}$$

$$a'_z = \frac{dV'_z}{d\tau}$$

$$= \frac{dV_z - (v/c)hdV_y}{1 - v^2/c^2 dt\sqrt{1 - v^2/c^2}}$$

$$= \frac{a_z - (v/c)\ a_y h}{(1 - v^2/c^2)^{3/2}}. \tag{17.71}$$

Next, using the new octonion $a' = a'_t h + a'_x i + a'_y j + a'_z k$ and $a = a_t h + a_x i + a_y j + a_z k$ to express accelerations, we examine whether acceleration is constant under coordinate transformation. From (17.68), (17.69), (17.70), and (17.71), we find that

$$|a'|^2 = (a'_t h + a'_x i + a'_y j + a'_z k)(a'_t h - a'_x i - a'_y j - a'_z k)$$

$$= -a'^2_t + a'^2_x + a'^2_y + a'^2_z$$

$$= \frac{1}{(1 - v^2/c^2)^3}\left\{-(v/c)^2 a_x^2 + a_x^2 + [a_y + (v/c)a_z h]^2 + [a_z - (v/c)\ a_y h]^2\right\}$$

$$= \frac{1}{(1 - v^2/c^2)^3}\left[-(v/c)^2 a_x^2 + a_x^2 + a_y^2 - (v/c)^2 a_z^2 + a_z^2 - (v/c)^2 a_y^2\right]$$

$$= \frac{1}{(1 - v^2/c^2)^3}(1 - v^2/c^2)(a_x^2 + a_y^2 + a_z^2)$$

$$= \frac{1}{(1 - v^2/c^2)^2}(a_x^2 + a_y^2 + a_z^2).$$

Since $a_t = dV_t/dt = 0$ from $V_t = cdt/dt = c$, the above equation becomes

$$|a'|^2 = \frac{1}{(1 - v^2/c^2)^2}(0 + a_x^2 + a_y^2 + a_z^2)$$

$$= \frac{1}{(1 - v^2/c^2)^2}(-a_t^2 + a_x^2 + a_y^2 + a_z^2)$$

$$= \frac{1}{(1 - v^2/c^2)^2}(a_t h + a_x i + a_y j + a_z k)(a_t h - a_x i - a_y j - a_z k)$$

$$= \frac{1}{(1 - v^2/c^2)^2}|a|^2.$$

Thus, we have

$$|a'| = \frac{|a|}{1 - v^2/c^2}, \tag{17.72}$$

which indicates that an acceleration is not invariant under coordinate transformation.

In special relativity, the acceleration as seen by the static system is also calculated using the proper time τ of a moving point mass. This is called four-acceleration, and is written as A. Four-acceleration is invariant under coordinate transformation like four-velocity. We explain this below. As explained in Section 17.4, four-velocity U_x, which is obtained by calculating all velocities using proper time τ, is $U_x = dx/d\tau$. Thus, the acceleration A, as seen from the static system is

$$\begin{aligned}
A_x &= \frac{dU_x}{d\tau} \\
&= \frac{d(dx/d\tau)}{d\tau} \\
&= \frac{d(dx/dt\sqrt{1 - v^2/c^2})}{dt\sqrt{1 - v^2/c^2}} \\
&= \frac{1}{1 - v^2/c^2} \times \frac{d(dx/dt)}{dt} \\
&= \frac{1}{1 - v^2/c^2} \times \frac{dV_x}{dt} \\
&= \frac{a_x}{1 - v^2/c^2}.
\end{aligned}$$

Similary, the other proper accelerations A_t, A_y, and A_z become

$$\begin{aligned}
A_t &= \frac{a_t}{1 - v^2/c^2}, \\
A_y &= \frac{a_y}{1 - v^2/c^2}, \\
A_z &= \frac{a_z}{1 - v^2/c^2}.
\end{aligned}$$

In addition, since a'_t, a'_x, a'_y, and a'_z are obtained using the proper time τ, we can write

$$A'_t = a'_t,\ A'_x = a'_x,\ A'_y = a'_y,\ A'_z = a'_z.$$

Thus, (17.72) becomes

$$|A'| = |A|. \tag{17.73}$$

However, A and A' are not vectors, but new octonions expressing four-acceleration. Therefore, four-acceleration is constant under coordinate transformation. The above result is in agreement with the result of special relativity. It indicates that a contradiction does not occur in calculations using the new octonion.

18

New Octonion and Mass

18.1 Mass is a time component of the unit world line

Newtonian mechanics is underpinned by two main assumptions: existence of stationary space and homogeneity of time. Movement is conceptualized in stationary space as a difference between points in the space. Moreover, time passes uniformly everywhere in space, and is identical for two or more observers. However, as Euclid wrote in *Elements*, if the rules of nature are divided into axioms and theorems, stationary space and homogeneous time are imprecise concepts. Although consistent with our experience, both premises are mathematical axioms rather than theorems. Because an axiom is experientially considered as correct and cannot be proven, it may be denied anytime.

Differing from Newton, Einstein wondered how light would appear to an observer traveling at velocity of light. Subsequently, he derived the Lorentz transformation, in which the time and distance of a point mass depend on the observer's velocity. Einstein questioned the status of stationary space and homogeneity of time as axioms. As explained in Section 13.1, questioning axioms leads to novel scientific concepts.

Einstein's relativity theory was seeded by his denial of stationary space, i.e., absolute space, and homogeneity of time, i.e., absolute time, assumed in Newtonian mechanics. However, he embraced the concept of absolute mass. He presumed the existence of mass itself, without querying its source. In Einstein's formulation, mass is treated as another physical quantity that depends on time t and distance x. Before mass and energy were equated in Einstein's famous formula, $E = mc^2$, they were related in the classical formula $E = mv^2/2$. Both formulations associate mass m, time t, and distance x through an equivalent relationship. Similarly, Einstein's formula

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}, \tag{17.11}$$

which expresses the velocity-dependent change in kinetic mass, does not elucidate the fundamental meaning of mass. In general relativity, mass is the pre-existent entity that bends space and creates gravity. In other words, relativity theory adopts the axiom that mass exists without questioning its origin. Thus, similar to Newton's acceptance of absolute space and absolute time, Einstein accepted absolute mass.

However, axioms tend to be improved throughout their history. In this section, we posit an axiom that mass is not an absolute quantity; rather, it is a time component of a unit world line. Subsequently, we investigate whether this axiom admits the velocity-dependent mass formulation of (17.11). In special relativity, since no contradiction is found, the energy is related to the time component p_t of the four-dimensional momentum multiplied by the velocity of light c. We test our axiom on this theory also.

Consider an elementary particle B (constituting substance) moving along the positive x-direction of a stationary observer A with uniform velocity v as shown in Figure 18.1. The four-dimensional space–time diagram in this situation is shown in Figure 18.2, where the y- and z-axes are omitted.

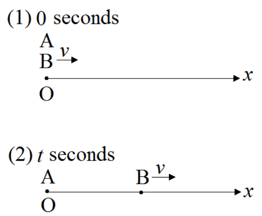


Figure 18.1

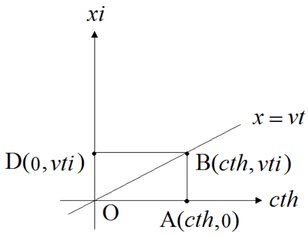


Figure 18.2

The line parallel to the cth-axis drawn from B intercepts the xi-axis at node D. The equation of the world line of B is $x = vt$. The new octonions of points A, B, and D are $A = cth$, $B = cth + vti$, and $D = vti$, respectively. The distance between the origin O and B is $|OB|$. If $c > v$, we find that

$$\begin{aligned}|OB| &= \sqrt{B\overline{B}} \\ &= \sqrt{(cth + vti)(cth - vti)} \\ &= cth\sqrt{1 - v^2/c^2}. \end{aligned} \tag{18.1}$$

The length of a unit world line is denoted by η (eta). However, it is a real number. When $|OB|$ is the length of a unit world line, the coordinates of B are assumed as $B_0(ct_0h,\ vt_0i)$. Since $|OB_0|$ in (18.1) is an imaginary number, $|OB_0|$ is $\eta \times h = \eta h$. From (18.1), we obtain

$$\eta h = ct_0h\sqrt{1 - v^2/c^2}.$$

Thus, we find that

$$t_0 = \frac{\eta}{c\sqrt{1 - v^2/c^2}}. \tag{18.2}$$

Let A_0 be the coordinates of A when $|OB|$ is the length of a unit world line. From (18.2), we find that

$$\begin{aligned} A_0 &= ct_0 h \\ &= \frac{\eta h}{\sqrt{1 - v^2/c^2}}. \end{aligned} \tag{18.3}$$

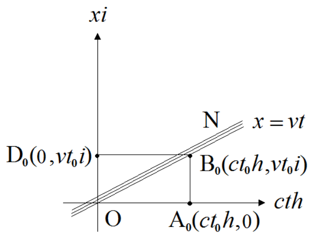


Figure 18.3

If the rest mass m_0 comprises N elementary particles B (see Figure 18.3), B traces N world lines. Hereafter, if B is associated with a single world line, it is referred to as an elementary particle, and if it is associated with N world lines, it is called a substance. From the axiom that mass is a time component of a unit world line, the rest mass m_0 can be obtained by multiplying N times the length $|OB_0| = \eta h$ of the unit time component of B (as seen from B) by a conversion factor δ (delta). Thus, we have

$$m_0 = \delta N \eta h. \tag{18.4}$$

The conversion factor δ achieves unit consistency between $N\eta h$ and the weight measured in grams (g). An analogous conversion factor normalizes the weight measured in UK (pounds) and the weight measured in USA (grams). The kinetic mass m is the product of δN and the time component A_0 of $|OB_0|$, which is observed by A. From (18.3), we find that

$$\begin{aligned} m &= \delta N A_0 \\ &= \frac{\delta N \eta h}{\sqrt{1 - v^2/c^2}}. \end{aligned} \tag{18.5}$$

From (18.4) and (18.5), we have

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

consistent with the formula of special relativity

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}. \tag{17.11}$$

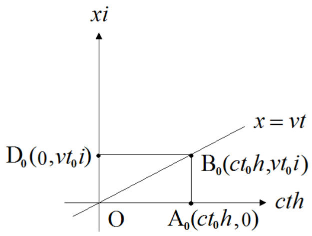


Figure 18.4

If the elementary particle B in Figure 18.4 travels at higher velocity v, the slope of the straight line $x = vt$ increases, and the denominator of the previously calculated formula

$$t_0 = \frac{\eta}{c\sqrt{1 - v^2/c^2}} \tag{18.2}$$

reduces. Thus, at higher velocities, time t_0 becomes greater and point $B_0(ct_0h,\ vt_0i)$, whose distance from the origin O is a unit world line, recedes from the origin O. This is a feature of four-dimensional space–time and beyond our common sense. Then, the time component $|OA_0|$ of $|OB_0|$ becomes greater. Since $|OA_0|$ expresses the kinetic mass m as seen from static observer A, increasing the velocity v increases the kinetic mass and intrinsically accounts for the increase of kinetic mass at higher velocities.

From (18.4) and (18.5), we find that mass is expressed by an imaginary number in our familiar positive world. Next, we reformulate the previous equations

$$\begin{aligned} |OB| &= \sqrt{B\overline{B}} \\ &= \sqrt{(cth + vti)(cth - vti)} \\ &= cth\sqrt{1 - v^2/c^2} \end{aligned} \tag{18.1}$$

in terms of the new octonion $B = ct + vthi$ in the negative world. If $c > v$, we find that

$$\begin{aligned} |OB| &= \sqrt{B\overline{B}} \\ &= \sqrt{(ct + vthi)(ct - vthi)} \\ &= \sqrt{c^2t^2 - v^2t^2} \\ &= ct\sqrt{1 - v^2/c^2}. \end{aligned} \tag{18.6}$$

Thus, mass is expressed by a real number in the negative world. Since time and distance in our positive world are expressed by imaginary numbers (as explained in Section 11.4), mass as the time component of a world line is also expressed by an imaginary number.

To express mass in our world by a real number, consider the positive and negative world points $ct + xhi + yhj + zhk$ and $cth + xi + yj + zk$, respectively. Assuming the former as our world point, the square of the distance in three-dimensional space becomes negative:

$$(xhi + yhj + zhk)(-xhi - yhj - zhk) = -x^2 - y^2 - z^2.$$

Since this result contradicts everyday observation, the world point in the positive world can reasonably be assumed as $cth + xi + yj + zk$ and it yields an imaginary mass.

If mass is the time component of a unit world line, η is related to the mass of an elementary particle, but is considered so short as to be unobservable. Indeed, η is comparable to the length of a fundamental string in string theory, as will be explained in Section 20.3. Since η is extremely small, it may be related to the Planck constant, an important basic quantity in quantum mechanics. Moreover, since elementary particles of different types extensively differ in mass, various categories of elementary particles should trace different world lines. As explained in Section 8.3, we posit an axiom that the world line of the particle is a wave as light, and that particles and light differ only by their paths in four-dimensional space–time. From that axiom, the wave amplitude Ψ becomes equivalent to the magnitude of a world line, and induces differences among particle masses. However, in the simple proof provided, Ψ is omitted from the calculations.

18.2 Energy and momentum

In Figure 18.3, δNA_0 expresses the kinetic mass m of the substance B.

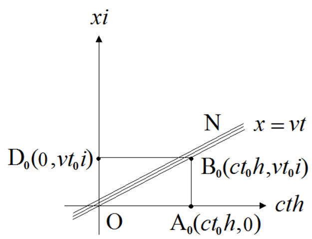


Figure 18.3

Applying the famous formula $E = mc^2$ of special relativity, the energy E of the substance is $\delta N A_0 c^2$. That is, we recover the theorem that energy is c^2 times the time component of a unit world line; however, we have also factored in the number N of world lines, the conversion factor δ, and the amplitude Ψ of a world line (to simplify the expression, Ψ has been omitted). This rule is a theorem because it is derived from the theorem $E = mc^2$ and the axiom that mass is a time component of a unit world line. The requirement for c^2 is examined in Section 18.4.

With no rationale except consistency, special relativity equates the product $p_t c$, of the time component of the four-dimensional momentum by c to the energy E. In special relativity, this is a vector quantity; however, it can be re-expressed in terms of the new octonion. Denoting the momentum components in the x-, y-, and z-directions as p_x, p_y, and p_z, respectively, the new octonion p of the momentum in four-dimensional space–time is

$$\begin{aligned} p &= p_t h + p_x i + p_y j + p_z k \\ &= (E/c)h + p_x i + p_y j + p_z k, \end{aligned} \tag{18.7}$$

where E, p_x, p_y, and p_z are real numbers. However, since we examine their imaginary number status in the following verifications, we transform (18.7) into

$$\begin{aligned} p &= p_t + p_x + p_y + p_z \\ &= (E/c) + p_x + p_y + p_z. \end{aligned}$$

In this equation, E, p_x, p_y, and p_z are not real numbers. Moreover, the momentum quantities p_t, p_x, p_y, and p_z involve the time t of observer A, different from the four-momentum calculated in proper time τ of the substance B.

Discerning readers of the book of special relativity may be perplexed by describing the energy E as the product of c and the time component p_t of the momentum in four-dimensional space–time. However, he/she could conceptualize mass as the time component of the world line in Figure 18.3 and energy as the same time component through the relationship $E = mc^2$. Such readers will also notice that the x-axial component $vt_0 i$ in Figure 18.3, if multiplied by δN, is equivalent to the x-component of the momentum p_x. However, the energy in a substance is mc^2 and the momentum p_x is $1/c \times E$ from (18.7). Thus we expect to obtain

$$p_x = \delta N v t_0 i \times c^2 \times 1/c = \delta N v t_0 c i.$$

This theorem states that the x-, y-, and z-components of the momentum are the products of c and the x-, y-, and z-axial components of a unit world line, respectively.

We prove this theorem as follows. Denoting the coordinates of D at the time when $|OB_0| = \eta$ as D_0, we have

$$D_0 = vt_0 i.$$

Thus, x-axial component summed over N world lines is

$$ND_0 = Nvt_0 i.$$

Then, we can rewrite

$$\begin{aligned} ND_0 &= Nvt_0 i \\ &= Nct_0 hvi/(ch). \end{aligned}$$

Multiplying both sides of this expression by the conversion factor δ, we find that

$$\delta ND_0 = \delta Nct_0 hvi/(ch).$$

Since mass is a time component of a unit world line, $\delta Nct_0 h$ is the kinetic mass m of substance B, and the equation becomes

$$\delta ND_0 = mvi/(ch). \tag{18.8}$$

Let us focus on a pertinent issue. Previously, we expressed the mass of a substance as

$$m = \frac{\delta N\eta h}{\sqrt{1 - v^2/c^2}}, \tag{18.5}$$

which contains the imaginary number h. Thus, if we apply the Newtonian definition of momentum, i.e., $p_x = mv$, we find that p_x is imaginary and therefore becomes a time component. Defining $p_x = mvi/h$, the imaginary number h in the mass m cancels and p_x contains the imaginary number i of the x-coordinate. Substituting $p_x = mvi/h$ in (18.8), we obtain

$$\delta ND_0 = p_x/c.$$

This formula transforms into

$$p_x = \delta ND_0 c.$$

The above expression is consistent with the theorem that momentum is c times the x-, x-, and x-axial components of a unit world line. However, to acquire the momentum of a substance, we must multiply by the number N of world lines and the conversion factor δ (omitting the amplitude Ψ of the world line).

18.3 The sum of mass or energy in four-dimensional space-time

The Universe is believed to have expanded from nothing during the Big Bang. According to this theory, the mass of the universe was initially zero. Although the nascent substance contained both positive and negative mass during the explosion, the positive and negative masses collided and canceled; however, the excess positive mass remained to form the Universe we know today. However, as explained in Section 18.1, the mass in our inhabited positive world is a positive imaginary quantity, while the mass in the overlapping negative world is a positive real quantity. The new octonion space–time theory admits only a positive imaginary mass and a positive real mass. Thus, whether this mass theory is compatible with the Big Bang theory is explored in this section.

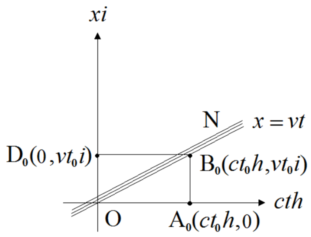


Figure 18.3

As shown in Figure 18.3, the kinetic mass m (denoted by $|OA_0|$) is a function of the velocity v of the substance B, for the following reason. Since the slope of the straight line $x = vt$ alters with velocity, the cth-axial component of a unit world line, i.e., the time component, changes. In contrast, the rest mass m_0 is the unit world line $|OB_0|$, which is fixed for all slopes of the straight line $x = vt$. That is, the mass residing in the Universe is the invariant rest mass m_0, while the kinetic mass m is the apparent mass, which may differ among observers.

In the reference frame of substance B, the coordinates of a point located at unit length ηh from the origin O along the world line of B are $(\eta h,\ 0,\ 0,\ 0)$ in the positive world and $(\eta,\ 0,\ 0,\ 0)$ in the negative world. Denoting the rest mass of B in the positive and negative worlds as m_0 and m_0' respectively, we have

$$m_0 = \delta N \eta h, \tag{18.4}$$

$$m_0' = \delta N \eta. \tag{18.9}$$

Recall that δ is a conversion factor. From (18.4) and (18.9), we find that

$$
\begin{aligned}
m_0^2 + m_0'^2 &= (\delta N \eta h)^2 + (\delta N \eta)^2 \\
&= -\delta^2 N^2 \eta^2 + \delta^2 N^2 \eta^2 \\
&= 0.
\end{aligned}
$$

That is, the squares of the rest masses in the positive and negative worlds sum to zero. Since mass and energy are equivalent, the squares of the energies also sum to zero. From this, we see that when summing quantities in the entire Universe, we must first square the quantity. If we sum the positive and negative energies, the calculation is easy. However, the Universe is not described by such simple values. Since this conclusion is obtained by the new octonion and the axiom that mass is a time component of a unit world line, we cannot determine whether the predicted value is consistent with that of the actual universe.

18.4 Meaning of c in $E = mc^2$

Some readers may seek a special semantic to c^2 in the mass/energy equivalence relation $E = mc^2$. Here, we prove that c^2 is required in calculations and we can express the equivalence relation as $E = m$. Unlike Newtonian mechanics, relativity theory allows interchange of time and distance units by changing t into ct. The momentum obtained by changing dt into cdt in the definition

$$
p_x = m\frac{dx}{dt}
$$

of Newtonian mechanics, defined as p_c, is called the new octonion momentum. Expressing the x-axial component of p_c as p_{cx}, we have

$$
p_{cx} = m\frac{dx}{cdt} = \frac{p_x}{c}. \tag{18.10}
$$

From (18.10) and the equation

$$
p_x = \delta N D_0 c
$$

which is proved in Section 18.2, we find that

$$
\begin{aligned}
p_{cx} &= \frac{p_x}{c} \\
&= \delta N D_0 c / c \\
&= \delta N D_0.
\end{aligned}
$$

From this equation, we see that D_0 in Figure 18.3 is the new octonion momentum of the elementary particle B (however, we must multiply by the number N of world lines and the conversion factor δ). Since we calculate in terms of time t, we need the light velocity c in the theorem that the x-, y-, and z-components of the momentum are the products of c and the x-, y-, and z-axial components of a unit world line. If the velocities are calculated from ct, the theorem describes the new octonion momentum as the x-, y-, and z-axial components of a unit world line.

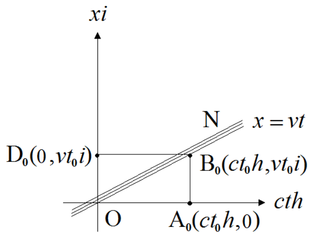


Figure 18.3

Similarly, we express the kinetic energy obtained by changing dt into cdt in the Newtonian kinetic energy formula

$$E = \frac{m}{2}(\frac{dx}{dt})^2$$

as E_c, called the new octonion kinetic energy. Now we have

$$E_c = \frac{m}{2}(\frac{dx}{cdt})^2 = \frac{E}{c^2}. \tag{18.11}$$

From (18.11) and $E = mc^2$ of special relativity, (18.11) becomes

$$E_c = \frac{mc^2}{c^2} = m.$$

That is, replacing by ct, the theorem that energy is c^2 times the time component of a unit world line is equivalent to stating that the new octonion energy is the time component of a unit world line, and is identical to mass.

The c^2 in $E = mc^2$ has no special semantic. Although the velocity could be calculated in terms of ct, it is habitually calculated in terms of t, with the factor c^2 explicitly stated in $E = mc^2$. If ct were to become an accepted unit, $E = mc^2$ would be rewritten as $E = m$ and mass, energy, and momentum could be illustrated on a four-dimensional space–time diagram.

The unit of a physical quantity is derived by a process called dimensional analysis. Applying dimensional analysis to ct, we have

$$ct = \frac{[distance]}{[time]}[time]$$
$$= [distance].$$

If we accept that mass is a time component of a unit world line, the mass unit also converts to a unit $[distance]$. In above scenario, the energy E and momentum p_x would have the same unit $[distance]$, rather than being related by $1/c$. Expressing quantities of time, distance, mass, energy, and momentum in the same unit $[distance]$ would remove the idea of unit.

18.5 Mass of light and the Higgs boson

In special relativity, light possesses no rest mass; however, it has kinetic mass and momentum. The idea of light acquiring mass only while it is moving cannot be visualized. However, if we use the axiom that mass is the time component of a unit world line, and construct a four-dimensional space–time diagram, the concept can be easily understood.

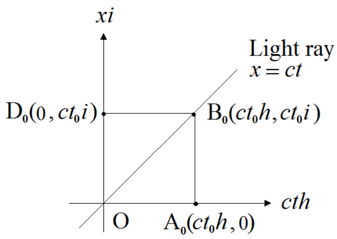


Figure 18.5

Figure 18.5 is the four-dimensional space–time diagram of light emitted in the positive x-direction of a stationary observer A. The y- and z-axes are omitted. The equation of the world line of light is $x = ct$. Substituting $x = ct$ into $cth + xi$, which expresses the coordinates of a point mass, the new octonion of light is given as $cth + cti$. Thus, the coordinates of point B on the world line of light are $(cth,\ cti)$ and light passes through all points that are equidistant from the cth- and xi-axes. The point located a unit world line ηh from the origin O is denoted $B_0(ct_0h,\ ct_0i)$. Assuming that mass is the time component of a unit world line, the rest mass m_0 and the kinetic mass m of light are given as $\delta\,|OB_0|$ and δA_0, respectively. The conversion factor δ changes the mass unit in four-dimensional space–time into the standard mass unit.

We now calculate the rest mass m_0 of light. From $B_0 = ct_0h + ct_0i$, we have

$$\begin{aligned} m_0 &= \delta\,|OB_0| \\ &= \delta\sqrt{(ct_0h + ct_0i)(ct_0h - ct_0i)} \\ &= \delta ct_0h\sqrt{1 - (ct_0i)^2/(ct_0h)^2} \\ &= \delta ct_0h\sqrt{1-1} \\ &= 0. \end{aligned}$$

That is, wherever point B_0 exists on the line $x = ct$, the length of light from the origin O is always zero. This is a feature of four-dimensional space–time and cannot be realized in three-dimensional space. Therefore, since light lacks a unit world line of distance ηh, its rest mass becomes zero.

What about the kinetic mass m? Figure 18.5 clearly shows the existence of a nonzero cth component of light; therefore, light possesses kinetic mass. However, since the point B_0 located at a distance ηh from the origin O does not settle, ct_0h is unfixed. Any particle other than light with finite mass and constant velocity possesses a fixed kinetic energy, given by $E = mv^2/2$ at nonrelativistic velocities. However, although light travels at constant velocity c, its energy is proportional to its frequency. More specifically, the energy of light E is related to its frequency ν (nu) through the Planck constant $\overline{h}$ as

$$E = \overline{h}\nu. \tag{18.12}$$

Although Planck's constant is correctly written as h, we here express it as $\overline{h}$ to distinguish it from h in the new octonion. As explained above, the kinetic mass m of light is the variable quantity δct_0h; thus, all kinetic masses (or energies) are permitted. This explains the proportionality between the energy of light and its frequency, given by (18.12), which admits all values, although the velocity of light is constant.

Assuming that energy is c^2 times the time component of a unit world line, the energy of light is,

$$\begin{aligned} E &= \delta ct_0 \times c^2 \\ &= \delta c^3 t_0. \end{aligned} \tag{18.13}$$

Here, the imaginary number h is omitted to avoid potential ambiguity with Planck's constant $\overline{h}$. From (18.12) and (18.13), we find that

$$\delta c^3 t_0 = \overline{h}\nu,$$

$$ct_0 = \frac{\overline{h}\nu}{\delta c^2}. \tag{18.14}$$

From (18.14), the cth-axial component A_0 of the unit world line of light of the frequency ν is $\overline{h}\nu/(\delta c^2)$. Moreover, from the theorem that momentum is c times the x-, y-, and z-axial components of a unit world line, if the space imaginary number i is omitted, the light of frequency ν has a momentum of

$$\begin{aligned} p &= \delta c t_0 \times c \\ &= \delta c^2 t_0. \end{aligned}$$

Substituting (18.14) into this equation, we have

$$p = \delta c \times \frac{\overline{h}\nu}{\delta c^2} = \frac{\overline{h}\nu}{c}.$$

This result is consistent with the result of special relativity.

We can also prove by calculation alone (without constructing a four-dimensional space–time diagram) that light has zero rest mass but unfixed kinetic mass, i.e., unfixed energy $\overline{h}\nu$. Inserting $v = c$ in the relationship between rest mass m_0 and the kinetic mass m, i.e.,

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}, \tag{17.11}$$

we have

$$m = \frac{m_0}{\sqrt{1 - c^2/c^2}} = \frac{m_0}{0},$$

which is meaningless unless $m_0 = 0$. That is, the rest mass of light is zero. Then, we have

$$m = \frac{0}{0}. \tag{18.15}$$

Rearranging (18.15) gives

$$m \times 0 = 0,$$

which admits any value of the kinetic mass m. The energy mc^2 of light is similarly unfixed. If $mc^2 = \overline{h}\nu$, the energy of light depends only on its frequency.

We now consider the Higgs boson. The Higgs boson, which supposedly gives particles their mass, has long been sought by elementary particle researchers. Immediately following the Big Bang, all particles are considered to have been massless and free-moving. Entities other than light gained mass only after their velocities were reduced by interaction with Higgs bosons.

If we admit the axiom explained in Section 8.3 that the world line of light differs from that of a substance only in its path in four-dimensional space–time, a point mass, which passes a point equidistant from the cth- and xi-axes in a coordinate plane, has velocity c, since its slope is c. Moreover, a point mass, which passes a point not equidistant from these axes, has velocity v, since its slope is not c. Because the world distance of a point on $x = ct$ is zero, light possesses no rest mass. The world distance of the point on $x = vt$, on the other hand, is non-zero; hence, substance has mass. That is, mass is not given; however, it is finite or zero depending on the locus in four-dimensional space–time. In this picture, no Higgs boson is required. However, if the world distance of all world lines was zero at the time of the Big Bang, all entities would have been massless light. If at that time the world lines of light and a Higgs boson collided, shifting the world distance of light from zero, a Higgs boson could be born. However, this scenario implies the pre-existence of the Higgs boson at the time of the Big Bang, which complicates the theory. Rather, we consider that the world lines of light and a substance differ only in their locus in the four-dimensional space–time, and that light possesses no rest mass because its world distance is zero. In contrast, a substance is characterized by a non-zero world distance, and hence has finite mass. This theory is simpler than the theory of the Higgs boson.

18.6 Constancy of the coordinate transformation of energy conservation

The energy conservation law states that the total energy of colliding bodies is unaltered by the collision. As proven in Section 17.6, the total momenta in the x-, y-, and z-directions are the same before and after a collision. Since this law holds true under a coordinate change, it is called a coordinate transformation invariance. Although standard relativity texts verify the coordinate transformation invariance of momentum, the same treatment for energy conservation is not found. We may naturally consider that total energy E is also invariant under coordinate transformation, since the energy E is the product of c and the time component p_t of the momentum, and the total momenta in the x-, y-, and z-directions are invariant under coordinate transformation. However, a mathematical demonstration of this phenomenon is lacking. This section proves that energy conservation is invariant under coordinate transformation using the new octonion.

Assume that two spheres A and B of rest masses m_0 and M_0 respectively collide with respective velocities v_0 and V_0, and move at respective velocities v and V after the collision. The situation is illustrated in Figure 18.6.

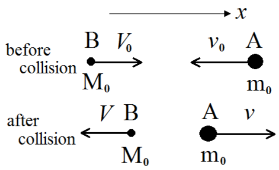


Figure 18.6

From the mass–energy relationship $E = mc^2$, and the relationship between rest mass and kinetic mass

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}, \tag{17.11}$$

the energy conservation law is written as,

$$\frac{m_0c^2}{\sqrt{1 - v_0^2/c^2}} + \frac{M_0c^2}{\sqrt{1 - V_0^2/c^2}} = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} + \frac{M_0c^2}{\sqrt{1 - V^2/c^2}}. \tag{18.16}$$

Equation (18.16) is formulated in the reference frame of a stationary observer. If we can prove (18.16) in a moving reference frame; for example, in the coordinates of sphere A moving at velocity v after the collision, then we can prove the coordinate transformation invariance of energy conservation.

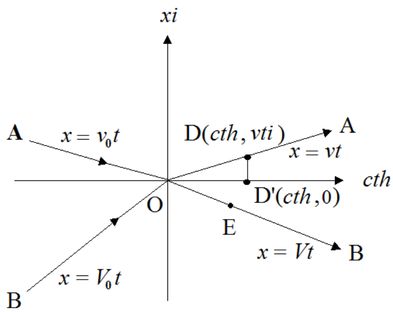


Figure 18.7

Expressing the collision as world lines, we obtain Figure 18.7. If the time of the collision is assumed as the origin O, the equations of the world lines of A and B before the collision are $x = v_0t$ and $x = V_0t$, respectively, and $x = vt$ and $x = Vt$, respectively, after the collision. Now, we implement a coordinate transformation to the coordinates of sphere A moving at velocity v after the collision. The point $D(cth,\ vti)$ on the world line $x = vt$ of A moves to point $D'(cth,\ 0)$ on the world line $x = 0$, i.e., the cth-axis. If this transformation can be expressed as a new octonion

H, given $D = cth + vti$ and $D' = cth$, we can write

$$(cth + vti)H = cth.$$

Rearranging this expression, we obtain

$$\begin{aligned} H &= \frac{cth}{cth + vti} \\ &= \frac{cth(cth - vti)}{(cth + vti)(cth - vti)} \\ &= \frac{cth(cth - vti)}{(cth)^2(1 - v^2/c^2)} \\ &= \frac{1 - vti/(cth)}{1 - v^2/c^2} \\ &= \frac{1 + vhi/c}{1 - v^2/c^2}. \end{aligned} \tag{18.17}$$

Next, the coordinates of the world line $x = Vt$ of sphere B are transformed through H. If A observes B moving at velocity V' after the coordinate transformation, the equation of B's world line (according to A) is $x' = V't'$. Since the coordinates of the point on the world line are $(ct'h,\ V't'i)$, we can write

$$ct'h + V't'i = (cth + Vti)H.$$

Substituting (18.17) into this equation, we find that

$$\begin{aligned} ct'h + V't'i &= (cth + Vti)\frac{1 + vhi/c}{1 - v^2/c^2} \\ &= \frac{cth - vti + Vti - Vvth/c}{1 - v^2/c^2} \\ &= \frac{(1 - Vv/c^2)cth + (V - v)ti}{1 - v^2/c^2}. \end{aligned}$$

Comparing the coefficients, we find that

$$\begin{aligned} t' &= \frac{(1 - Vv/c^2)t}{1 - v^2/c^2}, \\ V't' &= \frac{(V - v)t}{1 - v^2/c^2}. \end{aligned}$$

Eliminating t' from these equations gives

$$V' = \frac{V - v}{1 - Vv/c^2}.$$

That is, the equation $x' = V't'$ of the world line of B is

$$x' = \frac{(V - v)t'}{1 - Vv/c^2}. \tag{18.18}$$

Since V' is given by the Lorentz transformation of the velocity

$$\frac{V_x - v}{1 - (v/c^2)V_x} \tag{10.16}$$

as explained in Section 10.4, the coordinate transformation by the new octonion is verified.

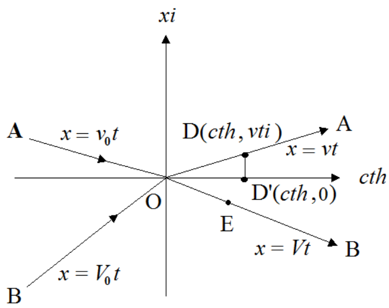


Figure 18.7

Next, we use (18.18) to derive the mass and energy of sphere B after the coordinate transformation. Denote the point located a unit distance ηh from the origin O on the world line (18.18) as $E'(ct'h,\ x'i)$. In Figure 18.7, point E moves to E' after the coordinate transformation. From

$$\begin{aligned} |\eta h|^2 &= |OE'|^2 \\ &= (ct'h + x'i)(ct'h - x'i), \end{aligned}$$

we have

$$-\eta^2 = -c^2t'^2 + x'^2. \tag{18.19}$$

Sustituting (18.18) into (18.19), we find that

$$\begin{aligned} -\eta^2 &= -c^2t'^2 + \frac{(V - v)^2t'^2}{(1 - Vv/c^2)^2} \\ &= \frac{-c^2t'^2(1 - Vv/c^2)^2 + (V - v)^2t'^2}{(1 - Vv/c^2)^2} \\ &= \frac{-c^2t'^2(1 - 2Vv/c^2 + V^2v^2/c^4) + (V^2 - 2Vv + v^2)t'^2}{(1 - Vv/c^2)^2} \\ &= \frac{-c^2t'^2 + 2Vvt'^2 - V^2v^2t'^2/c^2 + V^2t'^2 - 2Vvt'^2 + v^2t'^2}{(1 - Vv/c^2)^2} \end{aligned}$$

$$= \frac{-c^2t'^2 - V^2v^2t'^2/c^2 + V^2t'^2 + v^2t'^2}{(1 - Vv/c^2)^2}$$

$$= \frac{-c^2t'^2(1 - v^2/c^2) + V^2t'^2(1 - v^2/c^2)}{(1 - Vv/c^2)^2}$$

$$= \frac{-c^2t'^2(1 - v^2/c^2)(1 - V^2/c^2)}{(1 - Vv/c^2)^2}.$$

Conditional on $V < c$ and $v < c$, this equation becomes

$$c^2t'^2 = \frac{\eta^2(1 - Vv/c^2)^2}{(1 - v^2/c^2)(1 - V^2/c^2)},$$

$$ct' = \frac{\eta(1 - Vv/c^2)}{\sqrt{1 - V^2/c^2}\sqrt{1 - v^2/c^2}}. \tag{18.20}$$

Therefore, from (18.18) and (18.20), the new octonion expressing the point E' located at unit distance ηh from the origin O is

$$E' = ct'h + x'i$$

$$= ct'h + \frac{(V - v)t'i}{1 - Vv/c^2}$$

$$= \frac{\eta(1 - Vv/c^2)h}{\sqrt{1 - V^2/c^2}\sqrt{1 - v^2/c^2}} + \frac{\eta(V - v)(1 - Vv/c^2)i}{(1 - Vv/c^2)c\sqrt{1 - V^2/c^2}\sqrt{1 - v^2/c^2}}$$

$$= \frac{\eta(1 - Vv/c^2)h}{\sqrt{1 - V^2/c^2}\sqrt{1 - v^2/c^2}} + \frac{\eta(V - v)i}{c\sqrt{1 - V^2/c^2}\sqrt{1 - v^2/c^2}}.$$

This is a purely imaginary number whose first term (involving h) expresses kinetic mass and energy. The second term (involving i) expresses momentum as explained in Section 18.2. Thus, multiplying by the number N of world lines and the mass conversion factor δ, the kinetic mass of B observed by A after the coordinate transformation is

$$\frac{\delta N\eta(1 - Vv/c^2)h}{\sqrt{1 - V^2/c^2}\sqrt{1 - v^2/c^2}}. \tag{18.21}$$

Note that the amplitude Ψ of the world lines is not considered.

Suppose that sphere A traces n world lines. Since the rest mass of A before the collision is $m_0 = \delta n\eta h$, replacing N by n and V by v_0 in (18.21), the kinetic mass of A before the collision and after the coordinate transformation becomes

$$A(before): \ \frac{\delta n\eta(1 - v_0v/c^2)h}{\sqrt{1 - v_0^2/c^2}\sqrt{1 - v^2/c^2}} = \frac{m_0(1 - v_0v/c^2)}{\sqrt{1 - v_0^2/c^2}\sqrt{1 - v^2/c^2}}.$$

Similarly, since the rest mass of B before the collision is $M_0 = \delta N\eta h$, replacing V by V_0 in (18.21) gives the kinetic mass of B before the collision and after the coordinate transformation as

$$B(before):\ \frac{\delta N\eta(1-V_0v/c^2)h}{\sqrt{1-V_0^2/c^2}\sqrt{1-v^2/c^2}}=\frac{M_0(1-V_0v/c^2)}{\sqrt{1-V_0^2/c^2}\sqrt{1-v^2/c^2}}.$$

Now, replacing N by n and V by v in (18.21), the coordinate-transformed kinetic mass of A after the collision is

$$A(after):\ \delta n\eta h=m_0.$$

Since the coordinate-transformed kinetic mass of B after the collision is given by (18.21), we have

$$B(after):\ \frac{\delta N\eta(1-Vv/c^2)h}{\sqrt{1-V^2/c^2}\sqrt{1-v^2/c^2}}=\frac{M_0(1-Vv/c^2)}{\sqrt{1-V^2/c^2}\sqrt{1-v^2/c^2}}.$$

Multiplying each rest mass by c^2, the energy conservation law after the coordinate transformation is

$$\frac{m_0c^2(1-v_0v/c^2)}{\sqrt{1-v_0^2/c^2}\sqrt{1-v^2/c^2}}+\frac{M_0c^2(1-V_0v/c^2)}{\sqrt{1-V_0^2/c^2}\sqrt{1-v^2/c^2}}$$
$$=m_0c^2+\frac{M_0c^2(1-Vv/c^2)}{\sqrt{1-V^2/c^2}\sqrt{1-v^2/c^2}}. \tag{18.22}$$

If we can obtain (18.22) from the energy conservation law before the coordinate transformation, i.e.,

$$\frac{m_0c^2}{\sqrt{1-v_0^2/c^2}}+\frac{M_0c^2}{\sqrt{1-V_0^2/c^2}}=\frac{m_0c^2}{\sqrt{1-v^2/c^2}}+\frac{M_0c^2}{\sqrt{1-V^2/c^2}}, \tag{18.16}$$

and the momentum conservation law proven in Section 17.6, i.e.,

$$\frac{m_0v_0}{\sqrt{1-v_0^2/c^2}}+\frac{M_0V_0}{\sqrt{1-V_0^2/c^2}}=\frac{m_0v}{\sqrt{1-v^2/c^2}}+\frac{M_0V}{\sqrt{1-V^2/c^2}}, \tag{18.23}$$

then we can prove that energy conservation is invariant under a coordinate transformation. To this end, we attempt to derive (18.22) from (18.16) and (18.23). Dividing (18.16) by

$$\sqrt{1-v^2/c^2},$$

we obtain

$$\frac{m_0c^2}{\sqrt{1-v_0^2/c^2}\sqrt{1-v^2/c^2}}+\frac{M_0c^2}{\sqrt{1-V_0^2/c^2}\sqrt{1-v^2/c^2}}$$
$$=\frac{m_0c^2}{1-v^2/c^2}+\frac{M_0c^2}{\sqrt{1-V^2/c^2}\sqrt{1-v^2/c^2}}. \tag{18.24}$$

Multiplying (18.23) by

$$\frac{v}{\sqrt{1-v^2/c^2}}$$

gives

$$\frac{m_0 v_0 v}{\sqrt{1-v_0^2/c^2}\sqrt{1-v^2/c^2}} + \frac{M_0 V_0 v}{\sqrt{1-V_0^2/c^2}\sqrt{1-v^2/c^2}}$$
$$= \frac{m_0 v^2}{1-v^2/c^2} + \frac{M_0 V v}{\sqrt{1-V^2/c^2}\sqrt{1-v^2/c^2}}. \tag{18.25}$$

Subtracting (18.25) from (18.24), we find that

$$\frac{m_0 c^2(1-v_0 v/c^2)}{\sqrt{1-v_0^2/c^2}\sqrt{1-v^2/c^2}} + \frac{M_0 c^2(1-V_0 v/c^2)}{\sqrt{1-V_0^2/c^2}\sqrt{1-v^2/c^2}}$$
$$= \frac{m_0 c^2(1-v^2/c^2)}{1-v^2/c^2} + \frac{M_0 c^2(1-Vv/c^2)}{\sqrt{1-V^2/c^2}\sqrt{1-v^2/c^2}}.$$

This equation becomes

$$\frac{m_0 c^2(1-v_0 v/c^2)}{\sqrt{1-v_0^2/c^2}\sqrt{1-v^2/c^2}} + \frac{M_0 c^2(1-V_0 v/c^2)}{\sqrt{1-V_0^2/c^2}\sqrt{1-v^2/c^2}}$$
$$= m_0 c^2 + \frac{M_0 c^2(1-Vv/c^2)}{\sqrt{1-V^2/c^2}\sqrt{1-v^2/c^2}}.$$

This formula is identical to the energy conservation law obtained from the unit length ηh of the world line after a coordinate transformation, i.e.,

$$\frac{m_0 c^2(1-v_0 v/c^2)}{\sqrt{1-v_0^2/c^2}\sqrt{1-v^2/c^2}} + \frac{M_0 c^2(1-V_0 v/c^2)}{\sqrt{1-V_0^2/c^2}\sqrt{1-v^2/c^2}}$$
$$= m_0 c^2 + \frac{M_0 c^2(1-Vv/c^2)}{\sqrt{1-V^2/c^2}\sqrt{1-v^2/c^2}}. \tag{18.22}$$

The above derivations prove that the energy conservation law, given by

$$\frac{m_0 c^2}{\sqrt{1-v_0^2/c^2}} + \frac{M_0 c^2}{\sqrt{1-V_0^2/c^2}} = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} + \frac{M_0 c^2}{\sqrt{1-V^2/c^2}}, \tag{18.16}$$

is invariant after a coordinate transformation. That is, Equation (18.16) holds in any reference frame.

18.7 Force and a reflection of the world line

One of the first equations encountered by the high school student of mechanics is Newton's equation of motion

$$F = ma, \tag{18.26}$$

where F, m, and a denote force, mass, and acceleration, respectively. The accepted semantic of (18.26) is that a force F applied to a body of mass m induces an acceleration a. However, to what extent is this interpretation actually correct? Since F and ma are equated in (18.26), an applied force gives rise to an instantaneous acceleration (that is, the object accelerates in zero time). In special relativity, a signal cannot be instantaneously transmitted. In other words, a signal cannot be transmitted at infinite velocity. Thus, specifying the force F as the quantity that imposes an acceleration a on mass m contradicts the premise of special relativity.

Equation (18.26) is more properly interpreted in the following form:

$$ma = F. \tag{18.27}$$

This formula means that the status of a body of mass m having an acceleration a is a force F. In this interpretation, the force F, which imposes acceleration, does not exist but the status of ma is force. Force is merely a word which expresses the status of movement. In this way, the above contradiction is avoided, despite the equality of ma and F in (18.27).

To explain that force does not exist in four-dimensional space–time, we show the following example, in which the velocity changes in the absence of an applied force. Two steel balls of the same size and mass, traveling at v and $-v$ collide. After the collision, the balls reverse their direction, traveling at $-v$ and v, respectively. This collision is both symmetric and perfectly elastic (no energy loss). Now consider that, on collision, the velocity of the gravitational center of the steel ball instantaneously reverses from v to $-v$. Here, we need not consider the force exerted at the time of the collision. Rather, we may consider that the slope of the world line changes from v to $-v$. The force imparted during a collision is frequently called an action-reaction. However, no one can explain what it is.

From the above speculation, we here establish a new axiom that force does not exist, but is a status that world lines bend and reflect. If a world line continues bending after a collision, we say that force continues to act. If a body accelerates, its world line continues turning in the four-dimensional space–time diagram. Moreover, linear world line implies that no force is acting. In the world-line interpretation of force, we must establish the reflection law of world lines. When light is reflected by a plane mirror, the angle of reflection equals the angle of incidence. Recall our axiom that light and substance differ only by their locus in four-dimensional space–time. An analogous reflection law should therefore apply to the world lines of substances.

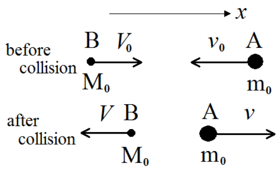


Figure 18.6

In Figure 18.6, the velocities v and V after the collision must be collectively determined from the velocities v_0 and V_0 before the collision. For this purpose, we require simultaneous equations in $v_0,\ V_0,\ v$, and V. The required equations are the energy conservation law

$$\frac{m_0c^2}{\sqrt{1-v_0^2/c^2}}+\frac{M_0c^2}{\sqrt{1-V_0^2/c^2}}=\frac{m_0c^2}{\sqrt{1-v^2/c^2}}+\frac{M_0c^2}{\sqrt{1-V^2/c^2}} \tag{18.16}$$

and the momentum conservation law

$$\frac{m_0v_0}{\sqrt{1-v_0^2/c^2}}+\frac{M_0V_0}{\sqrt{1-V_0^2/c^2}}=\frac{m_0v}{\sqrt{1-v^2/c^2}}+\frac{M_0V}{\sqrt{1-V^2/c^2}}, \tag{18.23}$$

which yield v and V from v_0 and V_0. Moreover, the reflection law of world lines must be invariant in any coordinates frame. In Sections 17.6 and 18.6, we proved that (18.16) and (18.23) are invariant under coordinate transformation. That is, since (18.16) and (18.23) satisfy two conditions required by the reflection law, they are considered as reflection laws of world lines.

Energy and momentum conservation are fundamental physical laws that are ubiquitously applied in physics and engineering problems. However, why these two laws exist is unresolved. The upper verification shows that energy and momentum conservation are merely reflection laws of world lines. Moreover, since many other physical laws can be proved by energy and momentum conservation, the reflection of world lines might explain many physical phenomena.

To conclude, mass and energy are both time components of a unit world line, while momentum is the space component of a unit world line. Force is a word which expresses the status of a world line. Furthermore, energy and momentum conservation laws are reflection laws of world lines. That is, only new octonions and world lines exist in four-dimensional space–time. All physical phenomena may be explained by reflections of world lines. Einstein interpreted gravity as the bending of space by mass. However, in the theory of the new octonion and world lines, mass is a time component of a unit world line, and does not bend space. As explained in Section 13.3, four-dimensional space–time is intrinsically bent, regardless of mass.

18.8 Extinction of the world line

In Section 17.6, we internally exploded a substance of rest mass M_0 into two substances, each of rest mass m_0 and velocities v and $-v$. Subsequently, we used momentum conservation to show that

$$M_0 = \frac{2m_0}{\sqrt{1 - v^2/c^2}}. \tag{17.62}$$

In this section, (17.62) is obtained from energy conservation and the axiom that mass is the time component of a unit world line. We also prove that a world line disappears and is transformed into energy.

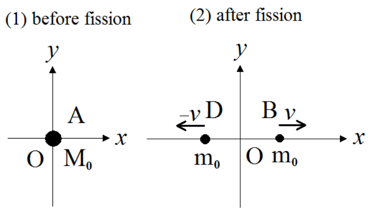


Figure 18.8

Figure 18.8 shows a substance A of rest mass M_0 located at the origin O. An interior explosion splits A into two substances B and D, each of rest mass m_0. The division occurs along the x-axis and substances B and D travel horizontally with velocities v and $-v$, respectively. The corresponding four-dimensional space–time diagram is shown in Figure 18.9.

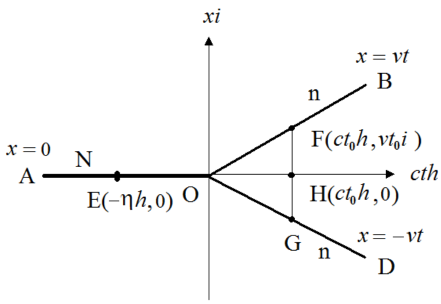


Figure 18.9

Since the substance A remains at rest, its distance x is zero. As time t passes, the world line $x = 0$ is generated for substance A. That is, substance A shifts along the cth-axis from the left to the origin O. If the fission occurs at the origin O in time, the world line of substance B after the fission satisfies $x = vt$ while that of substance D is given by $x = -vt$. On the straight lines $x = 0,\ x = vt$, and $x = -vt$, points $E,\ F$, and G are located one unit world line ηh from the origin O. The straight line FG intersects the cth-axis at node H. The coordinates of H are $(ct_0h,\ 0)$. Since the coordinates of F are $(ct_0h,\ vt_0i)$, it follows that

$$\begin{aligned}\eta h &= |OF| \\ &= \sqrt{(ct_0h + vt_0i)(ct_0h - vt_0i)} \\ &= ct_0h\sqrt{1 - (vt_0i)^2/(ct_0h)^2} \\ &= ct_0h\sqrt{1 - v^2/c^2}.\end{aligned}$$

Thus, we find that

$$ct_0 = \frac{\eta}{\sqrt{1 - v^2/c^2}}. \tag{18.28}$$

Assume that A traces N world lines before the fission, while B and D each trace n world lines after the fission. From $E = mc^2$ and the axiom that mass is the time component of a unit world line, the energy before and after the fission is $\delta N\,|OE|\,c^2$ and $2\delta n\,|OH|\,c^2$ respectively. Recall that δ is a mass conversion factor. Energy conservation gives

$$\delta N\,|OE|\,c^2 = 2\delta n\,|OH|\,c^2.$$

Thus, we have

$$N\,|OE| = 2n\,|OH|\,.$$

Substituting $|OE| = \eta h$ and $|OH| = ct_0h$ into this equation, we find that

$$N\eta = 2nct_0.$$

Substituting (18.28) into this equation, we have

$$N = \frac{2n}{\sqrt{1 - v^2/c^2}}. \tag{18.29}$$

Equation (18.29) relates the number N of world lines before the fission to the total number of world lines $2n$ after the fission. Rearranging (18.29), we obtain the number of world lines after fission as

$$2n = N\sqrt{1 - v^2/c^2}.$$

Since $\sqrt{1 - v^2/c^2} < 1$, the number of the world lines is reduced after the fission. This analysis reconfirms the proof in Section 17.6 that some of the world lines have been converted to the kinetic energy of the substances B and D.

Given that $E = mc^2$, (18.29), and accepting that mass is the time component of a unit world line, the rest energy before and after the fission differs by

$$\begin{aligned} M_0c^2 - 2m_0c^2 &= \delta N\eta hc^2 - 2\delta n\eta hc^2 \\ &= \frac{2\delta n\eta hc^2}{\sqrt{1 - v^2/c^2}} - 2\delta n\eta hc^2 \\ &\doteqdot 2\delta n\eta hc^2(1 + \frac{v^2}{2c^2}) - 2\delta n\eta hc^2 \\ &= 2\delta n\eta \frac{v^2}{2} h \\ &= 2m_0\frac{v^2}{2}h. \end{aligned} \tag{18.30}$$

The quantity $m_0v^2/2$ is only the Newtonian kinetic energy of substances B and D. Therefore, we have reconfirmed that the missing world lines after the fission are converted to kinetic energy. When a launched firework divides by explosion of the internal gunpowder, the world lines of the whole firework become fewer as some are converted to energy. When mitotic division occurs at near-light velocities, the result is nuclear fission, which releases vast amounts of energy.

18.9 Energy-momentum equation and Dirac's γ matrix

The energy–momentum equation is discussed in all special relativity texts. A stationary observer A views a point mass B of rest mass m_0 moving along a straight line with uniform velocity v. In A's reference frame, B carries momentum p and its energy E is

$$E^2 = (m_0c^2)^2 + c^2p^2, \tag{18.31}$$

where m_0c^2 is the rest energy of point mass B. However, p is not the four-momentum of special relativity but the momentum in the three-dimensional space of Newtonian mechanics. In this section, the energy–momentum equation (18.31) is derived in terms of the new octonion. We also derive a relationship between the new octonion and Dirac's γ (gamma) matrix in quantum mechanics.

As proved in Section 17.2, the momentum

$$mc\frac{dt}{dt}h + m\frac{dx}{dt}i + m\frac{dy}{dt}j + m\frac{dz}{dt}k \tag{18.32}$$

of point mass B in four-dimensional space–time is invariant under a coordinate transformation. In other words, the absolute values of (18.32) are identical in the reference frames of both A and B. However, since the momenta observed by A

and B are calculated in terms of time t and the proper time τ, respectively, they differ from the four-momentum of special relativity, expressed in terms of τ only. Although the mass is invariant in the four-momentum formulation, it depends on the velocity in (18.32).

First, we calculate the absolute value of the momentum in the coordinates of point mass B. Since B observes the proper time τ and is stationary in its own reference frame, $dx = dy = dz = 0$. Moreover, B's mass is the rest mass m_0. Therefore, the momentum p_B and the square of its absolute value $|p_B|^2$ observed by B are

$$\begin{aligned} p_B &= m_0 c \frac{d\tau}{d\tau} h + m_0 \frac{0}{d\tau} i + m_0 \frac{0}{d\tau} j + m_0 \frac{0}{d\tau} k \\ &= m_0 c h, \\ |p_B|^2 &= m_0^2 c^2 h^2 \\ &= -m_0^2 c^2. \end{aligned} \tag{18.33}$$

The momentum p_A of B and the square of its absolute value $|p_A|^2$ as observed by A are similarly calculated. In this reference frame, the time is t and the mass is the kinetic mass

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}.$$

Moreover, the momenta of the x-, y-, z-directions are p_x, p_y, and p_z, respectively. Thus, (18.32) becomes

$$\begin{aligned} p_A &= mc \frac{dt}{dt} h + m \frac{dx}{dt} i + m \frac{dy}{dt} j + m \frac{dz}{dt} k \\ &= mch + p_x i + p_y j + p_z k, \\ |p_A|^2 &= (mch + p_x i + p_y j + p_z k)(mch - p_x i - p_y j - p_z k) \\ &= -m^2 c^2 + p_x^2 + p_y^2 + p_z^2. \end{aligned} \tag{18.34}$$

Since momentum is invariant under coordinate transformation, we have

$$|p_B|^2 = |p_A|^2.$$

Substituting (18.33) and (18.34) into this equation, we find that

$$-m_0^2 c^2 = -m^2 c^2 + p_x^2 + p_y^2 + p_z^2.$$

Substituing $E = mc^2$ and the Newtonian momentum $p^2 = p_x^2 + p_y^2 + p_z^2$ into the above expression yields

$$\begin{aligned} -m_0^2 c^2 &= -E^2/c^2 + p^2, \\ E^2/c^2 &= m_0^2 c^2 + c^2 p^2. \end{aligned}$$

Multiplying both sides of this equation by c^2, we obtain

$$E^2 = (m_0c^2)^2 + c^2p^2.$$

That is, the energy–momentum equation (18.31) is recovered by the new octonion formulation.

Finally, we relate the new octonion to Dirac's γ matrix. Paul Dirac, who largely contributed to the development of quantum mechanics, factored the energy-momentum equation

$$E^2 = (m_0c^2)^2 + c^2p^2 \tag{18.31}$$

as follows:

$$\begin{aligned}\left(\frac{E}{c}\right)^2 &= p^2 + (m_0c)^2 \\ &= p_x^2 + p_y^2 + p_z^2 + (m_0c)^2 \\ &= (\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m_0 c)^2,\end{aligned}$$

$$\frac{E}{c} = \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m_0 c. \tag{18.35}$$

In this formulation, α_n and β are matrixes given by

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \ \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

α_n and β are related through the γ matrix as

$$i\beta\alpha_n = -\gamma_n.$$

Dirac incorporated the γ matrix into his classical wave equation for a free electron

$$\frac{i\beta}{c}\frac{\partial\Psi}{\partial t} - \left(\gamma_1\frac{\partial\Psi}{\partial x_1} + \gamma_2\frac{\partial\Psi}{\partial x_2} + \gamma_3\frac{\partial\Psi}{\partial x_3}\right) - \frac{m_0c}{\hbar}\Psi = 0.$$

Here, Ψ is a wave function and $\hbar$ is obtained by dividing Planck's constant h by 2π.

When deriving the energy–momentum equation using the new octonion, we specified

$$|m_0ch| = |mch + p_xi + p_yj + p_zk| .$$

Since $mc = E/c$ (rearranging $E = mc^2$), the above formula can be rewritten as

$$|m_0ch| = \left| \frac{E}{c}h + p_xi + p_yj + p_zk \right| .$$

Comparing this equation to

$$\frac{E}{c} = \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m_0 c, \tag{18.35}$$

we must question whether the new octonion relates to the γ matrix. In fact, Hiroyuki Kamada (Kyushu Institute of Technology) demonstrated that Dirac's γ matrix and the new octonion are mathematically equivalent (personal communication).

18.10 An easy method for obtaining $E = mc^2$

Here, we derive Einstein's famous special relativity formula $E = mc^2$ in terms of the new octonion. The derivation is very simple. In Newtonian mechanics, force, distance, and time are written as F, x, and t, respectively, and the energy E and momentum p are given as

$$E = Fx, \tag{18.36}$$

$$p = Ft. \tag{18.37}$$

Equations (18.36) and (18.37) can be added or subtracted, provided that their units are the same. To this end, we multiply both sides of (18.37) by the velocity of light c, and rewrite the equation as

$$cp = Fct. \tag{18.38}$$

The right-hand sides of (18.36) and (18.38) have identical units. Therefore, E and cp (or E/c and p), have identical units. Expressing the momentum in four-dimensional space–time in terms of the new octonion, we have

$$\begin{aligned} m\frac{d}{dt}(cth + xi + yj + zk) &= mc\frac{dt}{dt}h + m\frac{dx}{dt}i + m\frac{dy}{dt}j + m\frac{dz}{dt}k \\ &= mch + p_xi + p_yj + p_zk. \end{aligned} \tag{18.39}$$

As explained in Section 18.2, the time component mc of the momentum (18.39) in four-dimensional space–time is equivalent to energy. Thus, assigning the same unit to the time component mc of the momentum and the energy E through E/c, we find that

$$mc = E/c,$$
$$E = mc^2.$$

19

Special Relativity at the Superluminal Velocity

19.1 New Lorentz transformations at the superluminal velocity

When deriving the new Lorentz transformation in Section 3.5, it was explained that if we assume $v > c$, we obtain a new Lorentz transformation at the superluminal velocity. Here we calculate this transformation. Consider a stationary observer A at origin O and an observer B moving at uniform velocity v in the positive x-direction of A. In addition, a stationary observed point mass D is positioned at distance x from the origin. B and A coincide at the origin O at time $t = 0$. After t seconds, the coordinates of the three entities are $A(cth,\ 0)$, $B(cth,\ vti)$, and $D(cth,\ xi)$, and the corresponding complex numbers are $A = cth$, $B = cth + vti$, and $D = cth + xi$. The space–time diagram at the superluminal velocity is illustrated in Figure 19.1. Since $v > c$, the world line $x = vt$ of B leans toward the xi-axis from the world line of light $x = ct$.

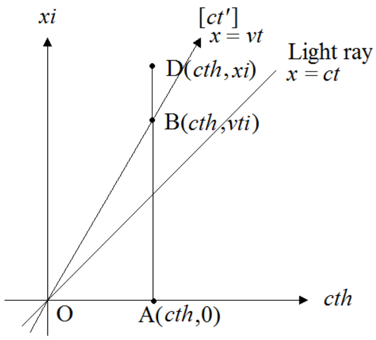


Figure 19.1

Calculating the coordinate transformation $D\overline{B}/|B|$ to obtain the new Lorentz transformation, we find that

$$\frac{D\overline{B}}{|B|} = \frac{(cth + xi)(cth - vti)}{\sqrt{(cth + vti)(cth - vti)}}$$
$$= \frac{(cth)^2 - cvt^2hi + xcthi - xvti^2}{\sqrt{(cth)^2 - (vti)^2}}.$$

From $v > c$ and $ct > 0$, the equation becomes

$$\frac{D\overline{B}}{|B|} = \frac{(cth)^2 - cvt^2hi + xcthi - xvti^2}{\sqrt{(cth)^2 - (vti)^2}}$$
$$= \frac{-(ct)^2 - cvt^2hi + xcthi + xvt}{\sqrt{-c^2t^2 + v^2t^2}}$$
$$= \frac{-(ct)^2 + xvt - cvt^2hi + xcthi}{\sqrt{c^2t^2(v^2/c^2 - 1)}}$$
$$= \frac{-(ct)^2 + xvt - cvt^2hi + xcthi}{ct\sqrt{v^2/c^2 - 1}}$$
$$= \frac{-ct + xv/c - vthi + xhi}{\sqrt{v^2/c^2 - 1}}$$
$$= \frac{c(v/c^2)x - ct + (x - vt)hi}{\sqrt{v^2/c^2 - 1}}. \tag{19.1}$$

Since hi is a real number, (19.1) is a purely real number. Therefore, the coordinate transformation transfers the world point of D to the negative world. Thus, if the new octonion of D as seen from B after the coordinate transformation is $D' = ct' + x'hi$, by coefficient comparison, we find that

$$ct' = \frac{c(v/c^2)x - ct}{\sqrt{v^2/c^2 - 1}},$$
$$t' = \frac{(v/c^2)x - t}{\sqrt{v^2/c^2 - 1}}, \tag{19.2}$$
$$x' = \frac{x - vt}{\sqrt{v^2/c^2 - 1}}. \tag{19.3}$$

Equations (19.2) and (19.3) specify the new Lorentz-transformed time and distance, respectively, at superluminal velocity.

We now examine the signs of (19.2) and (19.3). In Figure 19.1, when the point mass D is located above observer B, we have $x > vt$. At this time, since

$$x' = \frac{x - vt}{\sqrt{v^2/c^2 - 1}} > \frac{vt - vt}{\sqrt{v^2/c^2 - 1}} = 0$$

from (19.3), we find that $x' > 0$. On the other hand, when the point mass D lies below observer B, i.e., $x < vt$, we have

$$x' = \frac{x - vt}{\sqrt{v^2/c^2 - 1}} < \frac{vt - vt}{\sqrt{v^2/c^2 - 1}} = 0.$$

That is, $x' < 0$. Since the value of (19.2) is undefined when $x < vt$, we examine the case of $x > ct$. We have

$$t' = \frac{(v/c^2)x - t}{\sqrt{v^2/c^2 - 1}} = \frac{xv/c^2 - t}{\sqrt{v^2/c^2 - 1}} > \frac{vt/c - t}{\sqrt{v^2/c^2 - 1}} = \frac{t\sqrt{v/c - 1}}{\sqrt{v/c + 1}} > 0.$$

Thus, when the point mass D is located above the world line $x = ct$ of light, i.e., $x > ct$, t' is positive. Summarizing the above results, we obtain

$$x' > 0,\ t' > 0\ (\because\ x > vt),$$
$$x' < 0,\ t' > 0\ (\because\ vt > x > ct).$$

19.2 Proper time at the superluminal velocity

The equations

$$t' = \frac{(v/c^2)x - t}{\sqrt{v^2/c^2 - 1}}, \tag{19.2}$$

$$x' = \frac{x - vt}{\sqrt{v^2/c^2 - 1}} \tag{19.3}$$

obtained in the last section specify the time and distance of the point mass D as seen from observer B whose velocity exceeds the luminal velocity. Thus, we must examine the proper time of D moving at superluminal velocity with respect to B. For this purpose, we consider that observer B approaches D along the line AD, and specify $x' = 0$ when D coincides with B. Setting $x' = 0$ in (19.3), we obtain

$$0 = \frac{x - vt}{\sqrt{v^2/c^2 - 1}}$$

and $x = vt$. Since the time t obtained by substituting $x = vt$ into (19.2) is the proper time τ of D, we find that

$$\begin{aligned} \tau &= \frac{(v/c^2)vt - t}{\sqrt{v^2/c^2 - 1}} \\ &= \frac{t(v^2/c^2 - 1)}{\sqrt{v^2/c^2 - 1}} \\ &= t\sqrt{v^2/c^2 - 1}. \end{aligned} \tag{19.4}$$

Equation (19.4) is the proper time of the point mass moving at the superluminal velocity. Since $t > 0$, $\tau > 0$ and is therefore irreversible; a point mass D moving at superluminal velocity cannot return to the past. Hypothetical particles called tachyons, which move at superluminal velocities, have been suggested as the basis of time travel. However, the above calculations suggest that backward time travel is impossible. Moreover, since the world point of the point mass at the superluminal velocity is $ct' + x'hi$, which is a number in the negative world, the point mass exists in the negative world, and is therefore unobservable from the positive world. Whether a positive world substance can even exist in the negative world is unknown.

Furthermore, from (19.4), D's proper time τ passes more rapidly as the velocity v increases. This result is opposite to the time slowing effect as subluminal velocities approach c.

In determining Equation (19.4), we used the following two axioms, introduced in Section 13.2:

Axiom 5

The new octonion $\boldsymbol{B\overline{A}/|A|}$ is the coordinate transformation of $\boldsymbol{B}$ by $\boldsymbol{A}$ in four-dimensional space-time.

Axiom 7

The new octonion, describing the world point in four-dimensional space-time, is

$$\boldsymbol{A = ct_0 h + x_0 i + y_0 j + z_0 k + ct_1 + x_1 hi + y_1 hj + z_1 hk.}$$

However, $\boldsymbol{ct_0}$ and $\boldsymbol{ct_1}$ are both positive.

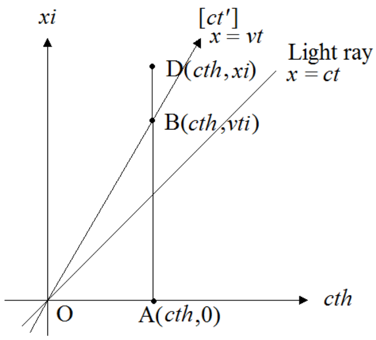


Figure 19.1

The space–time between the cth-axis and the straight line $x = ct$ in Figure 19.1 is our familiar subluminal space–time. The theorems and conclusions obtained from the above axioms in this space–time are entirely consistent. However, whether these axioms are realized in the space–time between the straight line $x = ct$ and the xi-axis is unknown. If Axioms 5 and 7 do not hold in space–time at superluminal velocities, the formulation of (19.4) may alter at these velocities.

19.3 Zeno's paradox and a discontinuity axiom

In the above calculations, we proved the possible existence of particles moving at superluminal velocities, because we can evaluate a proper time for these particles. However, c is generally regarded as an asymptotic speed; that is, the velocity v can approach that of light, but cannot exceed it. This issue can be resolved through Zeno's paradox.

Zeno was a Greek philosopher who lived in the 4th century B.C. Zeno's paradox takes various forms, but is typified by Achilles and the tortoise, which represent the swiftest running human being and one of the slowest moving animals, respectively. To present an intelligible argument, we place Achilles at the origin O and the tortoise slightly ahead of Achilles at distance x_0 along the x-axis. Achilles and the tortoise synchronously start to move. When Achilles arrives at the tortoise's original position x_0, the tortoise has moved to position x_1, ahead of its x_0. Later, when Achilles arrives at x_1, the tortoise has arrived at position x_2, ahead of position x_1. After repeating these movements several times, Achilles remains behind the tortoise. In reality, however, Achilles will rapidly catch up with the tortoise. Since theory and reality contradict, this situation is a paradox. Mathematically, such contradictions are treated by the theory of limits in calculus, expressed as lim (limit) or $\varepsilon - \delta$ (epsilon–delta). Intuitively, how an infinitesimal amount differs from zero is better explained by discussion rather than complicated mathematical theory.

As explained in Section 13.1, all theories have underlying axioms that can be examined to solve contradiction of the theory. The axiom of Zeno's paradox is that space and time are continuous quantities. Zeno considered that space and time are continuous with no gaps. In a continuous space–time, a quantity can be infinitesimal but not zero. If this axiom is redefined in discontinuous space–time, Zeno's paradox is no longer a paradox. We call this axiom of discontinuous space–time a discontinuity axiom.

To facilitate understanding, we consider Zeno's paradox in a different setting. Consider a river of width l that must be crossed, either by walking on a bridge or crossing on stepping stones. When a person uses the bridge to reach the opposite bank, he/she arrives halfway across the bridge (at $l/2$) at a certain time. Sometime later, the walker has moved a further $l/4$, half of the remaining distance to the opposite bank. If the walker continues to reduce his/her distance to the opposite bank by half, the distance to the opposite bank becomes infinitesimally small but never zero, and the person cannot cross the river. In calculus, this situation is resolved by taking the absolute limit

$$\lim_{n\to\infty}(\frac{1}{2})^n l = 0.$$

Intuitively, however, an infinitesimal amount cannot be exactly zero.

What happens if we use the stepping stones to reach the opposite bank? At some time, we reach a stone that is halfway between the two banks. Sometime later, we reach the last stepping stone that separates us from the opposite bank. Since the position halfway between this last stone and the opposite bank is in the river, we cannot move to this position. Instead, we jump to the opposite bank. If Achilles competes with the tortoise on stepping stones rather than on a continuous surface, he will eventually reach the tortoise's position and surpass it. Therefore, Zeno's paradox is not realized in discontinuous space. From elementary particle theory and quantum mechanics, space–time is known to be discontinuous at the very smallest scales. By the discontinuity axiom, each coordinate axis of our four-dimensional space–time is discontinuous. The continuity axiom has been adopted to comply with our everyday experience. Considering the discontinuity axiom, the velocity v of a particle may approach and reach the velocity of light c and eventually become superluminal. The axiom that space and time are discontinuous quantities is necessary for a complete understanding of four-dimensional space–time.

Here, the size of the discrete quantity becomes important. In Section 18.1, we specified the length of a unit world line as η. Since η expresses the mass of the smallest fundamental particle, from the axiom that mass is the time component of a unit world line, smaller lengths are impermissible. Thus, η may represent the unit length of discretized space–time.

20

Future Problems

20.1 Electromagnetism and biquaternion

Up to and including Chapter 19, we proved that many physical phenomena can be solved in terms of the new octonion. This chapter is devoted to currently unsolved problems, beginning with electromagnetism.

The basic laws of electromagnetism are the famous Maxwell field equations (named after James Maxwell.) Maxwell described the then-known rules of electromagnetism as twelve equations. Later, they were summarized to four fundamental equations. Specifically, the electric field $\boldsymbol{E}$, electric displacement $\boldsymbol{D}$, magnetic field $\boldsymbol{H}$, magnetic induction $\boldsymbol{B}$, current density $\boldsymbol{J}$, and volume density ρ (rho) of an electric charge are related by

$$\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} = \nabla \times \boldsymbol{H},$$

$$\nabla \cdot \boldsymbol{D} = \rho,$$

$$-\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \boldsymbol{E},$$

$$\nabla \cdot \boldsymbol{B} = 0.$$

Here, ∂ denotes a partial difference and ∇ (nabla) is specified in Cartesian coordinates as

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}.$$

In the above equations, $\boldsymbol{B}$, $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{J}$ are vectors and $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are not imaginary numbers of the new octonion, but unit vectors on the x-, y-, z-axes. The four Maxwell field equations are known to be invariant under the Lorentz transformation.

The Lorentz transformation derived from special relativity and the new Lorentz transformation derived from the new octonion yield the same formulae for t' and

x', but different formulae for y' and z'. In the traditional Lorentz transformation, $y' = y$ and $z' = z$. In Section 10.1, the new Lorentz transformation formulated by the new octonion was given as

$$y' = \frac{y + (v/c)zh}{\sqrt{1 - v^2/c^2}}, \tag{10.5}$$

$$z' = \frac{z - (v/c)yh}{\sqrt{1 - v^2/c^2}}. \tag{10.6}$$

As explained in previous chapters, these new y' and z' do not violate the constancy of the velocity of light, the world distance, momentum conservation law, or the dependence of mass on velocity. However, as explained in Section 16.5, if light is emitted perpendicular to the direction of an observer's movement, the result yielded by the new octonion differs from that of the Lorentz transformation. The correct formulation is indeterminable at present. Maxwell's electromagnetic field equations are certainly invariant under the standard Lorentz transformation. Their possibility of being invariant under the new Lorentz transformation remains an unanswered question. If invariance is proven, what are the y' and z' components of Maxwell's equations in the new octonion formulation? Applying the new Lorentz transformation to Maxwell field, equations may yield new solutions to phenomena such as synchrotron radiation. Proving invariance is the first obstacle in applying the new octonion to electromagnetism.

The identified problem mentioned above cannot be solved until Maxwell's field equations are appropriately expressed by the new octonion. As explained in Chapter 14, the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are related to their corresponding new octonions A and B by

$$B\overline{A} = \boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{B}, \tag{14.17}$$

$$\boldsymbol{A} \times \boldsymbol{B} = (B\overline{A} - A\overline{B})/2, \tag{14.26'}$$

$$\boldsymbol{A}\cdot\boldsymbol{B} = (B\overline{A} + A\overline{B})/2. \tag{14.27'}$$

In terms of these expressions, Maxwell's field equations might be reducible to simpler forms that are easier to understand. Moreover, when a vector is rewritten by a new octonion, its negative world component is easily identified. In the new octonion formulation, the negative world components are the real number components hi, hj, and hk. Thus, a portion of electromagnetic waves may reside in the negative world. Since the time of Hamilton, many researchers have attempted and failed to rewrite Maxwell field equations in terms of the quaternion. Thus, rewriting them by the new octonion is challenging.

We now turn to the third problem. As explained in Section 8.3, this book has accepted the axiom that the world line of a particle (constituting a substance) has the same properties as the world line of light. If the world line of an entity follows the path of zero world distance, where $s^2 = -c^2t^2 + x^2 + y^2 + z^2 = 0$, its velocity equals c, and it can be treated as light. Otherwise, the entity is called a substance because its velocity differs from c. Light is radiated as electromagnetic waves. If the world lines of particles and light differ only by their velocities, the world line of a substance may also satisfy Maxwell's field equations. Formulated in terms of the new octonion, Maxwell's field equations may convert v of the new octonion equations into c. In addition, the force calculated by the new octonion equations may be intrinsic to the force of the substance; that is, express its gravitational and nuclear force. Here lies the third problem in relating the new octonion to electromagnetism; can we find a new octonion equation that generalizes Maxwell's field equations? If such a new octonion equation exists, it may unify all known forces. Moreover, if the world lines of substances and waves can be considered equivalent, the substance world line is characterized by an amplitude Ψ (psi). The corresponding time component is $\partial\psi/\partial t$, the partial time differential of the wave function Ψ, which may be consistent with quantum mechanics. To establish this relationship, Maxwell's field equations must be rewritten in terms of the new octonion.

As the third edition of this book was being translated into English, it was revealed that Koen J. van Vlaenderen and Andre Waser reformulated Maxwell's field equations using biquaternions (double quaternions) in 2001. The biquaternion algorithm is identical to that of the new octonion. In other words, the new octonion and biquaternion are equivalent. Recall the algorithm presented in Section 3.4:

$$h^2 = i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, \ jk = -kj = i, \ ki = -ik = j,$$

$$hi = ih, \ hj = jh, \ hk = kh.$$

Moreover, the biquaternion A is obtained by re-expressing the coefficients of the quaternion $a + bi + cj + dk$ in complex number format as

$$A = (a + ph) + (b + qh)i + (c + rh)j + (d + sh)k.$$

Hamilton discovered the quaternion in 1843 and the biquaternion in 1844. The following 22 years of his research career were devoted to applying the quaternion to physics. However, he appears not to have appreciated the importance of the biquaternion. Hamilton published proof of the biquaternion in his *Lectures on*

Quaternions in 1853, nine years after its discovery, and discussed its applicability in his famous *Elements of Quaternions*. However, the biquaternion has been neglected in almost all quaternion texts to date.

Though the biquaternion and the new octonion are mathematically equivalent, their interpretation and application methods markedly differ in the following ways.

(1) In the biquaternion, h is treated as an attached imaginary number rather than a fourth imaginary number. Because the imaginary numbers i, j, and k form coordinate axes, they have both magnitude and direction, whereas h is considered as scalar. Thus, the algorithms of the biquaternion are

$$hi = ih,\ hj = jh,\ hk = kh.$$

The above relationships, where h is scalar, are assumed in all discussions of the biquaternion. On the other hand, in the new octonion, h is the fourth imaginary number and is more important than i, j, and k. As explained in Section 6.1, the important physical quantity of special relativity (proper time) is denoted by the imaginary number h. Mass and energy are also related through h, as explained in Chapter 18. Moreover, in Section 3.4, we found that inserting $hi = -ih$ in the new quaternion (or new octonion) does not yield the Lorentz transformation. Thus, in this book, we have assumed that $hi = ih$. On the other hand, the biquaternion allows $hi = ih$ because h is a scalar. Furthermore, $hi = ih$ indicates an important structure of four-dimensional space–time. Although the imaginary numbers i, j, and k are mutually related by $ij = k$, h is not mutually related to i, j, and k through $hi = ih$. Here, h expresses time and i, j, and k express space. Mathematically, therefore, we can move from one space to another, but not from space to time. That is, time travel is mathematically impossible.

(2) In the biquaternion, a number may be rewritten as follows:

$$\begin{aligned} A &= (a + ph) + (b + qh)i + (c + rh)j + (d + sh)k \\ &= (a + bi + cj + dk) + (p + qi + rj + sk)h. \end{aligned}$$

However, as explained in Section 11.4, the new octonion expresses the world point of a point mass in four-dimensional space–time. Thus, in the new octonion, a number is expressed as

$$\begin{aligned} A &= (ah + p) + (b + qh)i + (c + rh)j + (d + sh)k \\ &= (ah + bi + cj + dk) + (p + qhi + rhj + shk). \end{aligned}$$

The left (imaginary) and right (real) halves of the right-hand term express the world point in the positive and negative worlds, respectively. The new octonion, rather than the biquaternion, appears to be an ideal way of expressing world points in four-dimensional space–time.

(3) In the biquaternion formulation, h is not a formal imaginary number but an auxiliary number, and the biquaternion is a combination of quaternions. Thus, the biquaternion is considered as a member of the quaternion family. However, h is the most important component of the new octonion. In Section 11.8, hi, hj, and hk were identified as independent real numbers. Thus, the new octonion comprises four real and four imaginary numbers and strictly belongs to the octonion family. Since algebraic theory precludes more than eight dimensions, and the new octonion algebra describes curved four-dimensional space–time, it presents as an ultimate algebra. From this perspective, the new octonion is a more suitable mathematical formulation than the biquaternion.

(4) Biquaternion calculations are performed on vector quantities. Denoting the scalar components of two quaternions p and q by p_0 and q_0, respectively, and their respective vector components by $\boldsymbol{p}$ and $\boldsymbol{q}$, we have $p = p_0 + \boldsymbol{p}$ and $q = q_0 + \boldsymbol{q}$. Furthermore, if the inner product is $\boldsymbol{p}.\boldsymbol{q}$ and the outer product is $\boldsymbol{p} \times \boldsymbol{q}$, the two quaternions multiply as

$$pq = p_0 q_0 - \boldsymbol{p}.\boldsymbol{q} + p_0 \boldsymbol{q} + q_0 \boldsymbol{p} + \boldsymbol{p} \times \boldsymbol{q}.$$

This formula is widely used in quaternion and biquaternion calculations. As explained in Chapter 14, the vectors can be rewritten in terms of the new octonions as follows:

$$B\overline{A} = \boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{B}, \tag{14.17}$$

$$\boldsymbol{A} \times \boldsymbol{B} = (B\overline{A} - A\overline{B})/2, \tag{14.26'}$$

$$\boldsymbol{A}\cdot\boldsymbol{B} = (B\overline{A} + A\overline{B})/2. \tag{14.27'}$$

Moreover, as explained in Chapter 15, new octonions admit a wider range of calculations than tensors. Since complicated vector, tensor, and matrix calculations are simplified in the new octonion formulation, physical research may be undertaken purely by algebraic calculations in these formulations.

20.2 General relativity

In special relativity, an observer B moves along a straight line with uniform velocity v, relative to a stationary observer A. If observer B accelerates relative to observer A, his movement is described by general relativity. Succinctly, general relativity postulates that

(1) The velocity of light c depends on the acceleration a.

(2) Gravity results from the bending of space by mass.

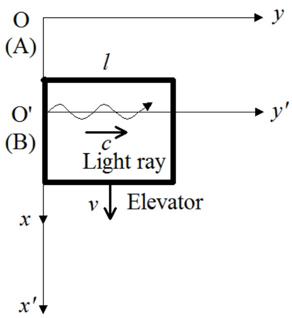


Figure 20.1

We now consider these postulates in terms of the new octonion. In general relativity texts, the first postulate is always exemplified by the case of a free-falling elevator. The pulley of an elevator breaks, sending the elevator into free-fall as shown in Figure 20.1. The falling direction is assumed as the positive x-axis and the horizontal direction is the y-axis. The coordinates of a stationary observer A are denoted $(x,\ y)$, while those of observer B inside the elevator are denoted by $(x',\ y')$. A and B are located at origins O and O', respectively. Moreover, at any given moment, the velocity of the elevator is v and B emits light in the y'-direction. B observes the width of the elevator as l_B.

First, we compute the Lorentz transformation of special relativity. The x-, y-, and z-axial components of the velocity of light, as seen by the stationary observer A, are denoted V_x, V_y, and V_z, respectively. These components are given by (see Section 16.2)

$$V_x = v, \tag{16.1}$$

$$V_y = c\sqrt{1 - v^2/c^2}, \tag{16.2}$$

$$V_z = 0. \tag{16.3}$$

Since the length along the y-axis is unaltered in the Lorentz transformation, the width l_B of the elevator observed by B is the same as that observed by A. Therefore, if light reaches the opposite wall of the elevator in time t_A as observed by A, we find that

$$t_A = \frac{l_B}{V_y} = \frac{l_B}{c\sqrt{1-v^2/c^2}}. \tag{20.1}$$

On the other hand, since $V'_y = c$, the time t_B taken for light to traverse the elevator in B's reference frame is given by

$$t_B = \frac{l_B}{c}. \tag{20.2}$$

Although the velocity v alters under acceleration, we consider its value at a particular instant. From (16.2), (20.1), and (20.2), the velocity of light emitted in the horizontal direction in the falling elevator is seen to be slower for a stationary observer A than for a moving observer B. The time at which light strikes the opposite wall is also longer from A's perspective. Because the infall velocity v increases under acceleration a, by (16.1) and (16.2), V_x increases while V_y decreases. As shown in Figure 20.2, A observes the path of light as downward parabolic.

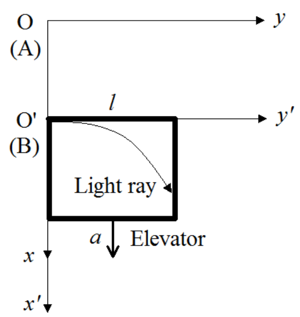


Figure 20.2

However, under the new Lorentz transformation, observer A records the y component of the velocity of light emitted in the y'-direction in the elevator as c. This result has been proven in Section 16.3. Under the new Lorentz transformation, we have

$$V_x = v, \tag{16.4}$$

$$V_y = c, \tag{16.7}$$

$$V_z = vh. \tag{16.8}$$

We now determine the times t_A and t_B at which light reaches the opposite wall of the elevator. Different from the traditional Lorentz transformation, the elevator width changes with velocity v in the new Lorentz transformation. We seek the relationship between the width l_B as seen from B and the width l_A as seen from A. The velocity of A relative to B is assumed as $-v$. At $z = 0$, setting $y = l_B,\ y' = l_A$, and $v_x = -v$ in the new Lorentz transformation

$$y' = \frac{y + v_x zh/c}{\sqrt{1 - v^2/c^2}}, \tag{10.5}$$

we find that

$$l_A = \frac{l_B}{\sqrt{1 - v^2/c^2}}.$$

Thus, from (16.7), we have

$$\begin{aligned} t_A &= \frac{l_A}{V_y} \\ &= \frac{l_B}{c\sqrt{1 - v^2/c^2}}. \end{aligned} \tag{20.3}$$

Moreover,

$$t_B = \frac{l_B}{c}. \tag{20.4}$$

Since (20.3) and (20.4) are identical to (20.1) and (20.2), the locus of light observed by A is that of Figure 20.2. However, in (20.1) and (20.2), the y component of c is $c\sqrt{1 - v^2/c^2}$, whereas in (20.3) and (20.4), it is exactly c. Although the y-axial components of the velocities are different in the two formulations, the travel times are identical because y' and y differ in the new and traditional Lorentz transformations.

From the above discussion, we find that, even when one reference frame is accelerating, the velocity of light along the y-direction is c in the new Lorentz transformation, and no contradiction occurs. Denoting the position by x and acceleration by a, general relativity gives the velocity of light c' in the moving reference frame as

$$c' = (1 + \frac{ax}{c^2})c. \tag{20.5}$$

The first problem in applying the new octonion to general relativity is to investigate its effect on (20.5).

We now examine the second postulate of general relativity; in that mass bends space, and thereby producing gravity. Einstein's theory of gravitation is wonderful because it correctly predicted the existence of black holes. However, mass being an independent physical quantity that bends space is debatable. As explained in Section 6.1, when special relativity (formulated in non-accelerating reference frames) is rewritten in terms of the new octonion, the space–time relationship intrinsically bends in the absence of mass. Moreover, in Chapter 18, we proved the special relativity result, according to which energy equals c times the time component of the momentum in four-dimensional space–time. We also demonstrated that, if mass is regarded as the time component of a unit world line, we recover the energy–mass equation $E = mc^2$. Einstein denied Newton's concept of absolute space and time and discovered that both quantities are relative. However, he accepted the concept of absolute mass. This book introduces an axiom that mass is the time component of a unit world line. If this axiom is accepted, absolute mass is denied and mass acquires a distance unit. Therefore, under the new octonion formulation, what is gravity? This constitutes the second problem in applying the new octonion to general relativity.

The third problem is whether the new octonion yields the Lorentz transformation in an accelerating system. The Lorentz transformation of special relativity has no equivalent transformation in an accelerating system in general relativity texts for the following reason. The Lorentz transformation is obtained using the constancy of the velocity of light, and is therefore inapplicable when the direction of the velocity v of a moving observer is arbitrary or when the reference frame accelerates. On the other hand, since we obtained the new Lorentz transformation using the coordinate transformation by the new octonion, the new Lorentz transformation may be obtainable in an accelerating system. The intermediate calculations are shown below.

Consider an observer B moving with proper acceleration a_0 on the *cth-xi* plane. A stationary observer A observes B accelerating with a. As explained in Section 17.7, a_0 and a are related by

$$a_0 = \frac{a}{(1 - v^2/c^2)^{3/2}}. \tag{17.69}$$

Since v is the velocity of B observed by A and $a = dv/dt$, we have

$$a_0 = \frac{dv/dt}{(1 - v^2/c^2)^{3/2}}.$$

Rearranging and integrating over dt, we find that

$$\int a_0 dt = \int \frac{dv}{(1 - v^2/c^2)^{3/2}},$$

$$a_0 t = \frac{v}{\sqrt{1 - v^2/c^2}} + C,$$

where C is the constant of integration. If the initial velocity is zero, i.e., $v = 0$ at $t = 0$, then $C = 0$, and we get

$$a_0 t = \frac{v}{\sqrt{1 - v^2/c^2}}. \tag{20.6}$$

Expressing (20.6) in terms of v, we obtain

$$\begin{aligned} a_0 t\sqrt{1 - v^2/c^2} &= v, \\ a_0^2 t^2 \left(1 - v^2/c^2\right) &= v^2, \\ a_0^2 t^2 &= v^2 + a_0^2 t^2 v^2/c^2, \\ a_0^2 t^2 &= v^2(1 + a_0^2 t^2/c^2), \\ v &= \frac{a_0 t}{\sqrt{1 + a_0^2 t^2/c^2}}. \end{aligned}$$

Note that $v > 0$ and $t > 0$. Since $v = dx/dt$, we can write

$$\frac{dx}{dt} = \frac{a_0 t}{\sqrt{1 + a_0^2 t^2/c^2}}.$$

Again rearranging and integrating over dt, we have

$$\int dx = \int \frac{a_0 t dt}{\sqrt{1 + a_0^2 t^2/c^2}},$$

which yields

$$x = \frac{c^2}{a_0}\sqrt{1 + (a_0 t/c)^2} + C', \tag{20.7}$$

where C' is a constant of integration. If $x = 0$ at $t = 0$, from (20.7), we have

$$\begin{aligned} 0 &= \frac{c^2}{a_0} + C', \\ C' &= -\frac{c^2}{a_0}. \end{aligned}$$

Substituting this result into (20.7), we find that

$$x = \frac{c^2}{a_0}\left[\sqrt{1 + (a_0 t/c)^2} - 1\right].$$

After some algebra, we obtain

$$\begin{aligned} \frac{a_0 x}{c^2} + 1 &= \sqrt{1 + (a_0 t/c)^2}, \\ \left(\frac{a_0 x}{c^2} + 1\right)^2 &= 1 + \left(\frac{a_0 t}{c}\right)^2, \end{aligned}$$

$$\left(\frac{a_0 x}{c^2}+1\right)^2-\left(\frac{a_0 t}{c}\right)^2=1. \tag{20.8}$$

Equation (20.8) describes the world line of observer B moving with proper acceleration a_0, and specifies the $ct'h$-axis.

According to (20.8), a curve that is linearly symmetric to the line $x = ct$ becomes the $x'i$-axis. The equation is solved by the method presented in Section 4.3. Interchanging x with ct; that is, replacing t by x/c in (20.8), we obtain the equation of the $x'i$-axis. Specifically, we have

$$\left(\frac{a_0 ct}{c^2}+1\right)^2-\left(\frac{a_0 x}{c^2}\right)^2=1,$$
$$\left(\frac{a_0 t}{c}+1\right)^2-\left(\frac{a_0 x}{c^2}\right)^2=1. \tag{20.9}$$

Equation (20.9) is the equation of the $x'i$-axis. (20.8) and (20.9) are hyperbolic curves.

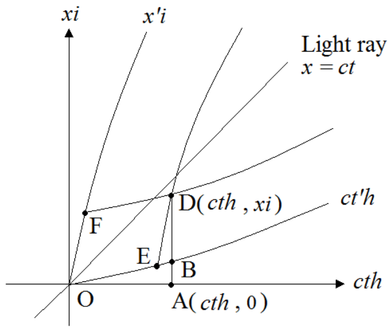


Figure 20.3

If the coordinates of a point mass D observed by A and B are $(cth,\ xi)$ and $(ct'h,\ x'i)$, respectively, the new complex plane becomes that of Figure 20.3. Allowing origin O to move along the curved $x'i$-axis, we perform a parallel translation of curve OB. The position of O when the curve passes point D is denoted by F. Similarly, if O moves along the curved $ct'h$-axis, we perform a parallel translation of curve OF. The position of O when the curve passes point D is E. At this instant, we have

$$ct'h + x'i = |OE|\, h + |OF|\, i.$$

Here, $|OE|$ and $|OF|$ are the lengths of the curves OE and OF, respectively. Together with (20.8), and (20.9), the above formula describes a coordinate transformation in an accelerating reference frame in terms of the new octonion. Integration must be used to obtain $|OE|$ and $|OF|$. We also require the theorem proved in Section 13.3, i.e.,

Theorem 26

The derivative world distance *dl* in two-dimensional space-time is

$$\boldsymbol{dl = chdt\sqrt{1 - (dx)^2/(cdt)^2}},$$

and (most likely) the hyperbolic functions $(e^x - e^{-x})/2$ and $(e^x + e^{-x})/2$. However, before performing the calculation, we must demonstrate that the exponential function e is realized in the *cth-xi* plane.

As shown in Figure 20.3, since the $ct'h$- and $x'i$-axes traveled by B in an accelerating system are curved, B observes that space–time is curved. Thus, even in the absence of mass, space–time is bent in an accelerating system. Importantly, the bending of space–time observed by B is relative. Absolute bending of space–time is restricted to the *cth-xi* and *yj*-$[zhk]$ planes in the new octonion formulation, as explained by Theorem 10 and Theorem 15 in Section 13.3. Obtaining the new Lorentz transformation in an accelerating system is the third problem in applying the new octonion to general relativity.

20.3 Five-dimensional space-time and string theory

String theory is one of the latest theories of modern physics. In this section, we consider the relationship between string theory and the new octonion space–time theory.

String theory evolved from the Kaluza–Klein theory. In the 1920s, physicists were seeking to combine Maxwell's field equations with the gravity theory of general relativity, into a so-called unified field theory. In 1919, Theodor Kaluza showed that Einstein's four-dimensional gravitational equation could be reformulated in five-dimensions. The resulting formula was found to embody both general relativity and electromagnetism. Moreover, in considering the physical meaning of this fifth dimension, Oskar Klein suggested that it exists as a very small round coil. The Kaluza-Klein theory was thought to have successfully united the electromagnetic and gravitational fields. However, the theory was abandoned by around 1940, because the fifth dimension was smaller than the size of a fundamental particle, and therefore unobservable.

Elementary particle physics studies the fundamental particles that constitute a substance. This complicated theory has successfully explained the properties of particles. On the other hand, in 1970, Yoichiro Nambu and his colleagues mathematically demonstrated that nuclear particles behave as oscillating one-dimensional strings of finite size, and the string theory was born. The first string theory required 26 dimensional space–time. However, in the supersymmetry theory of Pierre Ramond, every particle is coupled to a supersymmetric partner, reducing the number of space–time dimensions to 10. Furthermore, the extra dimensions are considered to be curled into small round structures, reminiscent of Kaluza–Klein theory.

The concepts of string theory are potentially consistent in the new octonion space–time theory. In the new octonion formulation, all particles are characterized by five variables; four variables $(ct,\ x,\ y,\ z)$ describing their location in four-dimensional space–time, and a variable Ψ indicating the amplitude of their world line. Moreover, as explained in Section 18.1, the size of the world line Ψ is smaller than a particle. In Kaluza–Klein theory, the predecessor of string theory, the fifth dimension was coiled into a structure that was too small to be observed. Brian Greene likened the curled up fifth dimension to the circumference of the cross section of a garden hose on the yard. The space–time theory of the new octonion similarly predicts a miniscule world line of an elementary particle with size Ψ. Thus, the fifth dimension of Kaluza-Klein theory may be identical to the amplitude Ψ of a world line.

Since all particles are described by five variables, namely, $ct,\ x,\ y,\ z$, and Ψ, five-dimensional space–time may appeal to physicists and mathematicians. However, since Ψ is the amplitude in four-dimensional space–time, space–time can only be realized in four dimensions. According to algebraic theory, a numerical system allowing the algebraic operations of addition, subtraction, multiplication, and division comprises only real numbers, complex numbers, quaternions, and octonions. Therefore, five-dimensional space–time is precluded on algebraic grounds.

Moreover, since the new octonion space–time theory predicts a doubly-structured four-dimensional space–time with overlapping negative and positive worlds, substances in the positive and negative worlds are each described by five variables, $(ct,\ x,\ y,\ z,\ \Psi)$ and $(ct',\ x',\ y',\ z',\ \Psi')$, respectively. Since supersymmetric string theory is realized only in ten-dimensional space–time, the ten variables of a substance in the new octonion space–time theory may be equivalent to ten-dimensional space–time. Although ten variables are specified, only doubly-structured four-dimensional space–time is recognized by the new octonion space–time theory.

String theory postulates that all substances are ultimately composed of extremely short vibrating strings. The new octonion space–time theory also accepts an axiom that the world line of a substance vibrates in a light-like manner. As explained in Section 19.3, space–time is discretized in the new octonion theory, and the length η

of a unit world line is smaller than a particle. Thus, the new octonion space–time theory shares many features of string theory, and the two theories may be fundamentally identical. However, whether string theory can be rewritten in terms of the new octonion is uncertain. Kaluza–Klein theory is a tensor theory. As explained in Section 15.2, tensor mathematics is limited to cross-sections of four-dimensional space–time, whereas the new octonion admits the entire four-dimensional space–time. However, the difference between tensor and the new octonion formulations is proven only in straight line coordinates. Whether the new octonion space–time theory is consistent with the curvilinear coordinates of general relativity has not been examined in this text.

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